Geometrical Aspects of Lie Groups and Homogeneous Spaces

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According to F. Klein's Erlangen program, the object of geoemtry is a G-space M, that is a set M with a given group G of transformations. If the groups acts transitively, that is for all $p, q \in M$ there exists an element in G which transforms p into q, then the G-space M is called homogeneous. As a result, if we pick any point $o \in M$, we can identify M with the set G/K of left cosets, where K is the subgroup of G consisting of those elements which map o to itself. Therefore, the homogeneous geometry of such a space $M = G/K$ is the study of those geometrical properties and those subsets of M, which are invariant under G.
Using the identification of a homogeneous space M with the quotient G/K, several geometrical problems can be reformulated in terms of the group G and the subgroup K. It particular, if G and H are Lie groups (i.e. manifolds with a group structure), the problem can be further reformulated in terms of their infinitesimal objects, i.e. the Lie algebra of G and its Lie subalgebra associated to K. The major benefit of such an infinitesimal approach is that difficult non-linear problems (from geometry, analysis or differential equations) can be reduced to linear algebra.
After Cartan's classification of semisimple Lie groups, two important classes of homogenous spaces were classified, namely symmetric spaces and flag manifolds. Flag manifolds are adjoint orbits of a compact semisimple Lie group, and equivalently homogeneous spaces of the form $G/C(S)$, where S is a torus in G. They have many applications in real and complex analysis, topology, geometry, dynamical systems, and physics.
The object of these lectures is to present introductory aspects of Lie groups and homogeneous spaces, as well as their geometrical objects defined on them such as invariant metrics and curvatures. A linear algebra background as well as an introductory course on differential manifolds and/or Riemannian geometry, should be more than sufficient prerequisites.

Here is an (ambitious) plan:

1. Lie Groups - examples 2. One-parameter subgroup 3. The adjoint representation and maximal tori 4. Classification of compact and connected Lie groups 5. Lie alebras 6. Left-invariant metrics and curvature 7. Homogeneous spaces - examples 8. Invariant metrics and curvature 9. Symmetric spaces and flag manifolds 10. Einstein metrics on homogeneus spaces 11. Fiber bundles 12. Homogeneous structures on fiber bundles 13. Applications to Yang-Mills Theory

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