Extremals of the generalized Euler-Bernoulli bending energy in real space forms and applications

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Classical **elasticae** (also known as **elastic curves**) are the solution to a variational problem proposed by **D. Bernoulli** to **L. Euler** in 1744, that of minimizing the bending energy of a thin inextensible wire. The mathematical formulation of this problem is that of minimizing the integral of the squared curvature of a curve γ, $\int_\gamma \kappa^2$, for curves of a fixed length satisfying given first order boundary data. When there is no penalty on the length, critical curves are known as **free elasticae**. **L. Euler** determined the plane elastic curves. Much later, **J. Radon** classified the free elastic curves in $\mathbb{R}^3$ in 1928.

One can extend the problem to any Riemannian manifold in an obvious manner. D. **Singer** and **J. Langer** have used classical techniques of the calculus of variations to derive the equations of the elasticae in a Riemannian manifold and to classify the closed free elastic curves in 2-dimensional space forms (1984) and in $\mathbb{R}^3$ (1985). In 1986 a different approach to this type of variational problem, using the theory of exterior differential systems, was developed by **R. Bryant and P. Griffiths**. A Hamiltonian approach using the machinery of Optimal Control Theory is due to **V. Jurdjevic** (1995).

We are interested in a generalization of the bending energy functional which was first considered (restricted to curves in $\mathbb{R}^3$) by **G.A. Bliss** (1907), **J. Radon** (1910) and by **W. Blaschke** (1930). We want to study the variational problem associated to curvature energy functionals of the type $F(\gamma) = \int_\gamma P(\kappa)$, acting on certain spaces of curves of a Riemannian manifold (as usual, here κ denotes the curvature function of γ and $P(\kappa)$ is a smooth function).

In these lectures, we will follow the ideas of **J. Langer** and **D. Singer** in order to describe a general method to deal with the above variational problem, paying special attention to closed curves in real space forms due to its special geometric significance. Critical points of these functionals will be referred to as **generalized elastic curves** and they have interesting applications to other higher dimensional variational problems in Physics, Biophysics and the theory of Submanifolds: elastic curves; fluid membranes and vesicles; elastic plates; models of relativistic particles; string theories and p-branes; Willmore surfaces; Chen-Willmore submanifolds; etc... We will review also some of these applications by studying particular choices of this energy in more detail.

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