Harmonic Spheres and Yang–Mills Fields

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ABSTRACT

In this lecture course we study a relation between harmonic spheres in loop spaces and Yang–Mills fields on the Euclidean 4-space.
Harmonic spheres are given by smooth maps of the Riemann sphere into Riemannian manifolds being the extremals of the energy functional given by Dirichlet integral. They satisfy nonlinear elliptic equations, generalizing Laplace–Beltrami equation. If the target Riemannian manifold is Kähler then holomorphic and anti-holomorphic spheres realize local minima of the energy. However, this functional usually have also non-minimal extremals.
On the other hand, Yang–Mills fields are the extremals of Yang–Mills action functional. Local minima of this functional are called instantons and anti-instantons. It was believed that they exhaust all critical points of Yang-Mills action on $\mathbb{R}^4$, until examples of non-minimal Yang-Mills fields were constructed.
There is an evident formal similarity between Yang–Mills fields and harmonic maps and after Atiyah's paper of 1984 it became clear that there is a deep reason for such a similarity. Namely, Atiyah has proved that the moduli space of G-instantons on $\mathbb{R}^4$ can be identified with the space of based holomorphic spheres in the loop space ΩG of a compact Lie group G. Generalizing this theorem, we formulate a conjecture that it should exist a bijective correspondence between the moduli space of Yang-Mills G-fields on $\mathbb{R}^4$ and the space of based harmonic spheres in the loop space ΩG.
In our lecture course we discuss this conjecture and an idea of its proof. All necessary notions from harmonic map and gauge field theories will be introduced in the course. We assume only basic knowledge of differential geometry and Lie group theory.

PROGRAM OF THE COURSE

I. HARMONIC SPHERES

1. Harmonic maps from the Riemann sphere into itself.
2. General definition of harmonic maps.
3. Harmonic spheres in Kähler manifolds.

II. INSTANTONS AND YANG–MILLS FIELDS

4. Yang–Mills action.
5. Instantons.

III. TWISTOR INTERPRETATION OF INSTANTONS

6. Basic twistor bundle over S^4.
7. Penrose twistor program and Atiyah–Hitchin–Singer construction.
8. Atiyah–Ward and Donaldson theorems.

IV. TWISTOR INTERPRETATION OF HARMONIC SPHERES

9. Eells–Salamon theorem.
10. Complex Grassmann manifolds and flag bundles.
11. Harmonic spheres in Grassmann manifolds: Burstall–Salamon theorem.

V. ATIYAH THEOREM AND HARMONIC SPHERES CONJECTURE

12. Loop spaces of compact Lie groups.
13. Holomorphic spheres in loop spaces: theorem of Atiyah.
14. Harmonic spheres conjecture.

VI. TWISTOR BUNDLE OVER THE LOOP SPACE

15. Hilbert–Schmidt Grassmannian.
16. Virtual flag bundles and twistor construction of harmonic spheres in Hilbert–Schmidt Grassmannian.
17. Embedding of loop spaces into Hilbert–Schmidt Grassmannian.

VII. IDEA OF THE PROOF OF HARMONIC SPHERES CONJECTURE

18. Harmonic analogue of Atiyah theorem.
19. Twistor interpretation of the moduli space of Yang–Mills fields.

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