

Group Analysis as a Microscope of Mathematical Modelling

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Abstract

Most of mathematical models in physics, engineering sciences, biomathematics, etc. are formulated as systems of nonlinear differential equations. Accordingly, engineering and science students and researchers routinely confront problems in mathematical modeling involving solution techniques for differential equations. Sometimes these solutions can be obtained analytically by numerous traditional ad hoc methods appropriate for integrating particular types of equations. More often, however, the solutions cannot be obtained by these methods, in spite of the fact that over 400 types of integrable second-order ordinary differential equations were summarized in voluminous catalogues.

On the other hand, the fundamental natural laws and technological problems formulated in terms of differential equations can be successfully treated and solved by Lie group methods. For example, Lie group analysis reduces the classical 400 types of equations to 4 types only! Development of group analysis furnished ample evidence that the theory provides a universal tool for tackling considerable numbers of differential equations even when other means of integration fail. In fact, group analysis is the only universal and effective method for solving nonlinear differential equations analytically. The old integration methods rely essentially on linearity as well as on constant coefficients. Group analysis deals equally easily with *linear and nonlinear* equations, as well as with constant and variable coefficients.

Lie group analysis can be called a microscope of mathematical modelling. Just like use of microscope in biology, group analysis allows to reveal symmetries of nonlinear mathematical models that cannot be seen otherwise. Numerous physical phenomena can be investigated using Lie symmetries to unearth various group invariant solutions and conservation laws that provide significant physical insight into the problem. An extensive compilation of the results of applications of Lie group analysis up to 1995 is given in three volumes [7].

The aim of this lecture course is to provide the wide audience of researchers and students with a comprehensive introduction to modern group analysis and to illustrate the advantages of Lie group analysis by various examples.

CONTENTS

1. Introduction to Lie group analysis
 - 1.1. Symmetry groups and Lie algebras of differential equations
 - 1.2. Integration of ordinary differential equations (ODEs) using symmetries. Four types of second-order ODEs
 - 1.3. Linearization of ODEs
 - 1.4. Linear ODEs reducible to algebraic equations
2. Group analysis of PDEs
 - 2.1. Invariant solutions. Applications in fluid mechanics and tumour growth model
 - 2.2. Group classification of irrigation systems
 - 2.3. Method of Euler-Laplace invariants for hyperbolic equations and its extension to parabolic equations
 - 2.4. Riemann's method from the group point of view
 - 2.5. Invariance principle in initial value problems
3. Nonlinear superposition principle
 - 3.1. Definition and main theorem for systems of ODEs
 - 3.2. Examples
 - 3.3. Integration using nonlinear superposition
4. Symmetries and conservation laws
 - 4.1. Noether's theorem
 - 4.2. Method of nonlinear self-adjointness
 - 4.3. Examples: gasdynamics, oceanology, nonlinear optics
 - 4.4. Use of conservation laws for solving nonlinear PDEs
5. Group theoretic modelling
 - 5.1. Determination of heat diffusion by Galilean group
 - 5.2. Propagation of light in curved space-times
 - 5.3. Group analysis of liquid metal systems
 - 5.4. Kepler's laws from the group point of view

References

- [1] S. Lie, Zur allgemeine Theorie der partiellen Differentialgleichungen beliebiger Ordnung, *Leipzig. Ber.*, vol. 1 (1895), 53–128. Reprinted in S. Lie, *Ges. Abhandl.*, Bd. 4, pp. 320–384. English translation “General theory of partial differential equations of an arbitrary order” is available in the book *Lie group analysis: Classical heritage*, ed. N.H. Ibragimov, ALGA Publications, Karlskrona, Sweden, 2004, pp. 1-63.
- [2] L. V. Ovsiyannikov, *Group properties of differential equations*, Siberian Branch, USSR Academy of Sciences, Novosibirsk, 1962 (in Russian).
- [3] L. V. Ovsiyannikov, *Group analysis of differential equations*, Nauka, Moscow, 1978. English transl., ed. W.F. Ames, Academic Press, New York, 1982.
- [4] N.H. Ibragimov, *Transformation groups applied to mathematical physics*, Nauka, Moscow, 1983. English transl. Reidel, Dordrecht, 1985.
- [5] P.J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York, 1986, 2nd ed. 1993.
- [6] G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New-York, 1989.
- [7] Ibragimov N.H., Ed. *CRC Handbook of Lie group analysis of differential equations* (CRC Press, Boca Raton); Vol. 1 (1994, 429 p), Vol. 2 (1995, 546 p.), Vol. 3 (1996, 536 p.).
- [8] N.H. Ibragimov, *Elementary Lie group analysis of ordinary differential equations*, John Wiley & Sons, Chichester, 1999.
- [9] N.H. Ibragimov, *A practical course in differential equations and mathematical modelling*, ALGA Publications, Karlskrona, 3rd ed. 2006. Now available: ISBN 978 981 4291 94-1, World Scientific & Imperial College Press, London, Singapore, 2009.
- [10] N.H. Ibragimov, Nonlinear self-adjointness in constructing conservation laws, *Archives of ALGA*, 7/8 (2010-2011) 1-99. See also *arXiv:1109.1728v1[math-ph]* (2011) 1-104.