Recent Developments On Ricci Solitons And Contact Geometry

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A Ricci soliton is a natural generalization of an Einstein metric, and is defined on a Riemannian manifold (M, g) by

$$(\pounds_V g)(X, Y) + 2Ric(X, Y) + 2\lambda g(X, Y) = 0$$

where $\pounds_V g$ denotes the Lie derivative of g along a vector field V, λ a constant, and arbitrary vector fields X, Y on M. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive respectively. Actually, a Ricci soliton is a generalized fixed point of Hamilton's Ricci flow [6]: $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. One may note here that Ricci flow was used by Perelman [10] to prove the celebrated Poincare's conjecture, and more generally, the Thurston's geometrization conjecture.

The vector field V generates the Ricci soliton viewed as a special solution of the Ricci flow, and would be called the generating vector field. A Ricci soliton on a compact manifold of dimensions 2 and 3 has constant curvature (see Hamilton [6] for dimension 2, and Ivey [9]). A Ricci soliton is said to be a gradient Ricci soliton, if $V = -\nabla f$ (up to a Killing vector field) for a smooth function f. A significant result of Perelman [10] says that a Ricci soliton on a compact manifold is a gradient Ricci soliton. For details, we refer to Chow et al. [3]. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [5]), and have been studied within the frame-work of general relativity in Reddy-Sharma-sivaramakrishnan [11].

In [12], Sharma initiated the study of Ricci solitons as Riemannian metrics associated to a contact structure. By a contact structure we mean a globally defined 1-form η on a smooth manifold M (of odd dimension $2n+1$) such that $\eta \wedge (d\eta)^n \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ (Reeb vector field) such that $d\eta(\xi, X)= 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X,Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. For details we refer to the classic monograph [1]. A contact metric structure is said to be K-contact if ξ is Killing. The contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler. For a Sasakian manifold, the restriction of φ to the contact sub-bundle D $(\eta = 0)$ is denoted by J and $(D, J, d\eta)$ defines a Kaehler metric on D, with the transverse Kaehler metric g^T related to the Sasakian metric g as $g = g^T + \eta \otimes \eta$. The transverse Ricci tensor Ric^T of g^T is given by

$$Ric^T(X,Y) = Ric(X,Y) + 2g(X,Y)$$

for arbitrary vector fields X, Y in D. The Ricci form ρ and transverse Ricci form ρ^T are defined by

$$\rho(X,Y) = Ric(X, \varphi Y), \quad \rho^T(X,Y) = Ric^T(X, \varphi Y)$$

for $X, Y \in D$. If the basic first Chern class $2\pi c_1^B$ of D, represented by ρ^T, vanishes, then the Sasakian structure is said to be null (transverse Calabi-Yau). We refer to Boyer, Galicki and Matzeu[2] for details. A contact metric manifold M is said to be η-Einstein in the wider sense, if the Ricci tensor can be written as

$$Ric(X,Y) = \alpha g(X,Y) + \beta \eta(X) \eta(Y)$$

for some smooth functions α and β on M. We know that α and β are constant if M is K-contact, and has dimension greater than 3.

A D-homothetic deformation $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\varphi} = \varphi, \bar{g} = ag + a(a-1)\eta \otimes \eta$ for a positive constant a, transforms a K-contact η-Einstein manifold into a K-contact η-Einstein manifold such that $\bar{\alpha} = \frac{\alpha + 2 - 2a}{a}$ and $\bar{\beta} = 2n - \bar{\alpha}$. The particular value: $\alpha = -2$ remains fixed under a D-homothetic deformation.

Definition 1 *A K-contact η-Einstein manifold with $\alpha = -2$ is said to be D-homothetically fixed.*

A simple example is a Sasakian space-form with constant φ-sectional curvature -3, identifiable with a $(2n+1)$-dimensional Heisenberg group.

In [12] Sharma obtained the following result.

Theorem 1 *A complete K-contact gradient Ricci soliton is compact Einstein and Sasakian.*

This was also shown later independently by He and Zhu [8] for the Sasakian case. In [7], Ghosh, Sharma and Chow proved the following result.

Theorem 2 *If a non-Sasakian (k,μ)-contact metric manifold (a generalization of Sasakian manifold and the trivial sphere bundle $E^{n+1} \times S^n(4)$) is a gradient Ricci soliton, then it is locally flat in dimension 3, and locally isometric to $E^{n+1} \times S^n(4)$ in higher dimensions.*

Subsequently, Cho and Sharma obtained the following two results.

Theorem 3 *A compact contact Ricci soliton with a generaing vector field point-wise collinear with the Reeb vector field, is Einstein.*

Theorem 4 *A homogeneous gradient contact Ricci soliton whose Reeb vector field is an eigenvector of the Ricci tensor, is locally isometric to $E^{n+1} \times S^n(4)$.*

Recently, Sharma and Ghosh [13] obtained the following result.

Theorem 5 *A 3-dimensional Sasakian metric which is a non-trivial Ricci soliton, is homothetic to the standard Sasakian metric on the Heisenberg group nil^3.*

Most recently, Ghosh and Sharma have extended the foregoing result to higher dimensions and answered the following question of H.-D. Cao (cited in [8]): "Does there exist a shrinking Ricci soliton on a Sasakian manifold, which is not Einstein?" in the negative by obtaining the following characterization and classification result.

Theorem 6 *If the metric of a $(2n+1)$-dimensional Sasakian manifold M (η, ξ, g, φ) is a non-trivial Ricci soliton, then (i) M is null η-Einstein (i.e. D-homothetically fixed and transverse Calabi-Yau), (ii) the Ricci soliton is expanding, and (iii) the generating vector field V leaves the structure tensor φ invariant, and is an infinitesimal contact D-homothetic transformation.*

They have also obtained the following result related to the preceding result.

Theorem 7 *If an η-Einstein contact metric manifold M admits a vector field V that leaves the structure tensor φ and the scalar curvature invariant, then either V is an infinitesimal automorphism, or M is D-homothetically fixed and K-contact.*

Proofs of the last three theorems use the formulas for the commutation between Lie-derivatives and co-variant derivatives of metric tensor, Levi-Civita connection, curvature tensor, Ricci tensor and scalar curvature, as given in Yano [14].

Remark: Note that a Ricci soliton as a Sasakian metric is different from the Sasaki-Ricci soliton in the context of transverse Kaehler structure in a Sasakian manifold, for example see Futaki et al. [4]).

This talk will conclude with some open promising questions on this topic for further developments.

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