Relativistic Hyperbolic Geometry

Abraham A. Ungar
Department of Mathematics
North Dakota State University
Fargo, North Dakota, USA
Email: Abraham.Ungar@ndsu.edu

Hyperbolic geometry was introduced by Lobachevsky (1829) and by Bolyai (1832) as a counterintuitive geometry that denies the Euclid's postulate according to which there exists in the plane only one line parallel to a given line through a given point not on the line. Several decades later, Einstein introduced his special theory of relativity in 1905. This physical theory is counterintuitive as well since, for instance, it implies that velocity addition is, generally, neither commutative nor associative.

The counterintuitive hyperbolic geometry of Lobachevsky and Bolyai, and the counterintuitive special relativity of Einstein were discovered independently. However, they met each other in 1908 when Varičak discovered that special relativity has natural interpretation in hyperbolic geometry.

In fact, we will see that when hyperbolic geometry and special relativity meet, they cross-pollinate to produce a novel way to study these two disciplines under the same umbrella. Indeed,

(1) Einstein addition law of relativistically admissible velocities encodes novel algebraic structures called a *gyrogroup* and a *gyrovector space*.
(2) The resulting Einstein gyrovector spaces form the algebraic setting for hyperbolic geometry, just as vector spaces form the algebraic setting for Euclidean geometry. As such, they enable Cartesian and barycentric coordinates to be introduced into hyperbolic geometry. The mathematical tools that Cartesian and barycentric coordinates provide, commonly used in the study of Euclidean geometry, can now be used in the study of hyperbolic geometry as well.
(3) Being the geometry that underlies special relativity, hyperbolic geometry, now equipped with Cartesian and barycentric coordinates, improves our way to study special relativity, thus demonstrating the cross-fertilization of special relativity and hyperbolic geometry at work.

We employ the algebra that Einstein relativistic velocity addition law encodes to enrich, enliven, and enhance the study of both special relativity theory and its underlying hyperbolic geometry. The resulting relativistic hyperbolic geometry is the mathematical discipline that studies the Albert Einstein (1879–1955) relativistic velocity model of the hyperbolic geometry of Nikolai Ivanovich Lobachevsky (1792–1850) and János Bolyai (1802–1860).

The sparkling beauty of Einstein's special theory of relativity manifests itself when it is placed in the framework of the hyperbolic geometry of Lobachevsky and Bolyai, giving rise to *relativistic hyperbolic geometry.* The hyperbolic space of relativistic hyperbolic geometry is the c-ball $\mathbb{R}^n_c$,

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}, \tag{1}$$

of the Euclidean n-space $\mathbb{R}^n$. In physical applications $n = 3$, but in geometry $n \geq 1$ is any positive integer.

Einstein's special relativity stems from his addition law of relativistically admissible velocities that he introduced in his 1905 paper that founded the theory. The resulting Einstein addition, $\oplus$, is a binary operation in the c-ball $\mathbb{R}^n_c$ of relativistically admissible velocities, which takes the vectorial form

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}\,, \tag{2}$$

where (i) $c > 0$ is a constant that, when $n = 3$, represents the speed of light in empty space, (ii) the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ are n-dimensional relativistically admissible velocities, (iii) $\gamma_\mathbf{v}$ is the gamma factor of special relativity,

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}\,, \tag{3}$$

and (iv) $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm that the c-ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$. Einstein addition in the c-ball $\mathbb{R}^n_c$ thus gives rise to pairs $(\mathbb{R}^n_c, \oplus)$ known as Einstein *gyrogroups*. In Einstein gyrogroups we define $\ominus\mathbf{v} = -\mathbf{v}$, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{v}\oplus(-\mathbf{v}) = \mathbf{0}$, $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$, $\ominus\mathbf{u}\oplus\mathbf{v} = (-\mathbf{u})\oplus\mathbf{v}$ and $\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v}$.

In the non-relativistic limit, when the speed of light c approaches infinity, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ and ordinary vector addition, $+$, in $\mathbb{R}^n$ coalesce.

Here we have to remember that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. In 1905, Einstein calculated the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2) of Einstein addition.

Einstein addition underlies the Lorentz transformation of special relativity theory. Being neither commutative nor associative, Einstein addition, $\oplus$, is seemingly structureless, as opposed to the Lorentz transformation of special relativity, which enjoys the algebraic structure known as a *group*. As a result, the pristine clarity of Einstein addition is obscured behind the cloud of Lorentz transformation. Einstein's intuition was, therefore, left dormant for about 80 years until it was brought back into a new mathematical life in 1988 in the author's article: *"The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities"* [4].

Being non-associative, Einstein addition gives rise to *gyrations* gyr$[\mathbf{u}, \mathbf{v}]$. For each pair $(\mathbf{u}, \mathbf{v})$ the gyration gyr$[\mathbf{u}, \mathbf{v}]$ is, in general, a non-trivial map of $\mathbb{R}^n_c$, given by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\}\,, \tag{4}$$

for all relativistically admissible velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n_c$. Note that in the special case when $\oplus$ is associative, the map gyr$[\mathbf{u}, \mathbf{v}]$ is trivial, so that gyr$[\mathbf{u}, \mathbf{v}]$ measures the extent to which $\oplus$ deviates from associativity.

Surprisingly, gyrations regulate Einstein addition in an elegant way, giving rise to the following laws and properties for all relativistically admissible velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \in \mathbb{R}_c^n$:

$$
\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}\\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Left Reduction Property}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Right Reduction Property}\\
\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Even Property}\\
(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} &= \mathrm{gyr}[\mathbf{v},\mathbf{u}] && \text{Gyration Inversion Law}\\
\mathbf{a}\cdot\mathbf{b} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} && \text{Inner Product Gyroinvariance}\,.
\end{aligned}
\tag{5}
$$

The reduction properties of gyrations in (5) trigger a remarkable reduction in complexity. Following the algebraic properties in (5), Einstein addition can be interpreted as a peculiar vector addition in the c-ball $\mathbb{R}_c^n$, whose departure from commutativity and associativity is controlled by gyrations which, in turn, possess their own rich structure.

The coincidence involved in the gyrocommutative-gyroassociative laws of Einstein addition in (5) gives rise to the gyroalgebra that we use extensively. This coincidence is amazing, compelling one to ask: why? How can it be that the same gyration, $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, that repairs the breakdown of commutativity in Einstein addition, repairs the breakdown of associativity in Einstein addition as well? Seeing the gyrocommutative-gyroassociative laws for the first time is like watching a magician pull a rabbit out of a hat. After studying the resulting gyrogroups and gyrovector spaces since 1988 [4], the author still has that reaction. Indeed, we will see that the mere introduction of gyrations turns Euclidean geometry into hyperbolic geometry, where Einstein addition is regulated by gyrations, playing the role of vector addition. Accordingly, Einstein addition is the relativistic analog of vector addition, which is more complex than vector addition, but much richer in structure.

In the resulting gyroalgebra, we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix "gyro" stems from "gyration", which is the mathematical abstraction of the special relativistic effect known as "Thomas precession". The resulting group-like structure to which Einstein addition gives rise is thus naturally called a *gyrocommutative gyrogroup.*

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to Einstein *gyrovector spaces.* The latter, in turn, form the algebraic setting for the relativistic model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

Various studies related to special relativity, gyrovector spaces and their underlying hyperbolic geometry are found, for instance, in [1, 7, 12]. Novel results in relativistic hyperbolic geometry that stem from the author's related books [5, 6, 8, 9, 10, 11], [13], will be presented.

References

[1] Azniv K. Kasparian and Abraham A. Ungar. Lie gyrovector spaces. *J. Geom. Symm. Phys.*, 1(1):3–53, 2004.

[2] Th. M. Rassias. Book Review: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, by Abraham A. Ungar. *Nonlinear Funct. Anal. Appl.*, 13(1):167–177, 2008.

[3] Th. M. Rassias. Book Review: *A gyrovector space approach to hyperbolic geometry*, by Abraham A. Ungar. *J. Geom. Symm. Phys.*, 18:93–106, 2010.

[4] Abraham A. Ungar. The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities. *Appl. Math. Lett.*, 1(4):403–405, 1988.

[5] Abraham A. Ungar. *Beyond the Einstein addition law and its gyroscopic Thomas precession: The theory of gyrogroups and gyrovector spaces*, volume 117 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 2001.

[6] Abraham A. Ungar. *Analytic hyperbolic geometry: Mathematical foundations and applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[7] Abraham A. Ungar. The proper-time Lorentz group demystified. *J. Geom. Symm. Phys.*, 4:69–95, 2005.

[8] Abraham A. Ungar. *Analytic hyperbolic geometry and Albert Einstein's special theory of relativity*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[9] Abraham A. Ungar. *A gyrovector space approach to hyperbolic geometry*. Morgan & Claypool Pub., San Rafael, California, 2009.

[10] Abraham A. Ungar. *Barycentric calculus in Euclidean and hyperbolic geometry: A comparative introduction*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[11] Abraham A. Ungar. *Hyperbolic triangle centers: The special relativistic approach*. Springer-Verlag, New York, 2010.

[12] Abraham A. Ungar. Gyration: The missing link between classical mechanics with its underlying euclidean geometry and relatlvisitc mechanics with its underlying hyperbolic geometry. In *P.M. Pardalos and Th.M. Rassias (eds.): Essays in Mathematics and its Applications. In honor of Stephen Smale's 80th birthday.*, pages 463–504. Springer, New York, 2012.

[13] Scott Walter. Book Review: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, by Abraham A. Ungar. *Found. Phys.*, 32(2):327–330, 2002.