Lie and Conditional Symmetries of Boundary Value Problems: Definitions, Algorithms and Applications to Physically Motivated Problems

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Nowadays Lie symmetries are widely applied to study partial differential equations (including multidimensional PDEs), notably, for their reductions to ordinary differential equations (ODEs) and constructing exact solutions. There are a huge number of papers and many excellent books (see, e.g., the book [3] and papers cited therein) devoted to such applications. Over recent decades, other symmetry methods, which are based on the classical Lie method, were derived. The Bluman-Cole method of non-classical symmetry (other widely used terminology is Q-conditional symmetry) is most well-known among them and the book [3] summarizes the results obtained by means of this approach for scalar PDEs. In recent papers [4, 5], this method are essentially extended and applied to non-linear PDE systems (including multi-component case).

However, a PDE (a system of PDEs) cannot model any real process without additional condition(s) on the unknown function(s). On the other hand, boundary-value problems (BVPs) based on the relevant PDEs (systems), which reflect general physical laws, describe many real processes arising in nature and society. One may note that the symmetry-based methods were not widely used for solving BVPs. The obvious reason follows from the following observation: the relevant boundary and initial conditions are usually not invariant under any transformations, i.e., they don't admit any symmetry of the governing PDE(s). Nevertheless, there are some classes of BVPs which can be solved by means of the Lie symmetry based algorithm. Such algorithm uses the notion of Lie's invariance of BVP in question. Probably, the first rigorous definition of Lie's invariance for BVPs was formulated by Bluman in 1970s [2] (the definition and several examples are summarized in the book [3]). This definition was used (explicitly or implicitly) in several papers to derive exact solutions of some BVPs. It should be noted that Ibragimov's definition of BVP invariance [9], which was formulated independently, is equivalent to Bluman's. On the other hand, one notes that Bluman's definition does not suit to all types of boundary conditions. Notably, the definition doesn't work in the case of boundary conditions involving points at infinity (e.g., $x \to \infty$) and conditions on the moving surfaces such as the Stefan conditions.

King [10] was probably the first who noted the problem with points at infinity and suggested the appropriate substitution to transform such points to regular those. Pukhnachov [11] and, independently, Benjamin and Olver [1] proposed how to define Lie's invariance on the moving boundaries in the case of some BVPs arising in hydrodynamics.

In our recent papers (see [6] and references therein), a new definition of Lie's invariance of BVP with a wide range of boundary conditions (including those on non-regular manifolds and moving surfaces) was formulated. Moreover, an algorithm of the group classification for the given class of BVPs was worked out for the first time. In [7] these results were extended in the multidimensional case. The definition and algorithm were applied to some classes of nonlinear two-dimensional and multidimensional BVPs of Stefan type with the aim to show their efficiency. In particular, the group classification problem for these classes of BVPs was solved, reductions to BVPs of lower dimensionality were constructed and examples of exact solutions(with physical meaning) were found.

Finally, there are many realistic BVPs, which cannot be solved using any definition of Lie's invariance of BVP, hence, definitions involving more general types of symmetries should be worked out. In [8], a new definition of conditional invariance for boundary-value problems (BVPs) is proposed. Its relation with the definitions, which were earlier worked out for Lie's invariance, is shown and the wider applicability is demonstrated.The definition is applied for reduction of BVPs with the basic reaction-diffusion-convection equation, which is used in physically and biologically motivated problems. The constrains on the functions arising in the boundary conditions are established and the nonlinear BVPs in question are thus reduced to those based on linear second-order ODEs. Moreover, the exact solution of a nonlinear BVP with zero Neumann conditions is found.

The lecture will be based on the papers [6, 7, 8] and some unpublished results.

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