Group-Theoretical Models of Internal and Collective Degrees of Freedom. Bertrand Systems and Their Generalizations

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1 Affinely Rigid Body and Affine Invariance in Physics

It was a surprising fact that the two apparently completely different objects were described by the same person: Leonhard Euler. And that the corresponding dynamical equations are now called "Euler equations". Nobody was then aware of the theory of Lie groups. But it seems that Euler intuitively felt the corresponding relationship between two problems. Indeed, today we know that the rigid body and the ideal fluid are two special, in a sense opposite, cases of dynamical systems on groups. In a sense "on Lie groups", although in the case of ideal fluids the term "Lie group" is misused. First of all one can ask the question why the dynamical systems with Lie groups as configuration spaces are so physically distinguished. There are two points for that: so-to speak theoretical and practical ones. From the very theoretical point of view it is clear that every system on a Lie group is distinguished by the fact that there are Lie groups which rule the spatial and internal geometry of degrees of freedom. And because of this the corresponding group becomes relevant for this problem. The corresponding Lie groups seem to organize and put in a hierarchic way the dynamics of degrees of freedom. But, what is also not to be neglected, any Lie group is a system ruled with analytic functions of dynamical variables. This analytical structure seems to suggest that all physically important results may be obtained explicitly in terms of analytical functions, first of all, in terms of analytic functions known from the realm of special functions theory. In a sense, this denotes the "direct solvability". Rigid body mechanics is based on the kinetic energy (metric tensor) invariant under the space-like rigid rotations. This is the left-hand-side invariance. At the same time, there is a hierarchy of right-invariance depending on the symmetries of the co-moving inertial tensor. At the same time, the right-hand side invariance under all volume-preserving diffeomorphisms is a characteristic feature of the infinite-dimensional Hamiltonian system describing the ideal fluid. In our approach to affinely-rigid body there is a finite-dimensional configuration space GL(n,R) which also admits some finite-dimensional group of left-acting and right-acting transformation groups. An important problem which appears then is the total left- or right- invariance of the kinetic energy. It turns out that unlike the rigid body case there is no affine invariance of the total kinetic energy (the spatial metric tensor). But in spite of this one can construct the left- or right- invariant kinetic energies. It is interesting that the resulting geodetic Hamiltonian systems can describe elastic vibrations. In a sense this resembles the Maupertuis variational principle which collects the dynamics in the form of kinetic energy (dynamical metric tensor). It is interesting that in spite of the non-compactness of the configuration space there are both bounded (quantum: discrete) and decaying solutions. Similar results may be obtained for systems on the unitary groups and probably for the complex linear group. Our model is something "between" the rigid body and ideal fluid. It admits deformative degrees of freedom, but in a finite-dimensional sense. Analytically, we use the polar and two-polar decomposition. This means that formally the problem is reduced to the two interacting metrically-rigid bodies coupled through the deformation invariants. An interesting feature is the dissociation threshold appearing in geodetic problems; of course, one can modify it by the isotropic potential. This problem appears both on the classical and quantum level. We have also described the related problem on U(n). It is interesting what will result in the complex case GL(n,C). There exist of course the matrix-exponential solutions of doubly-affinely invariant problems. There is for some reason the interest in one-side affine and one-side metrical invariance. It may be shown that there are also similar solutions, but they are subject to certain restrictions (stationary solutions).

2 Bertrand Systems on Spaces of Constant Sectional Curvature. The Action-Angle Analysis. Classical, Quasi-Classical and Quantum Problems

The classical Bertrand systems are those concerning the motion of material point in the spherically-symmetric potential force. The are such that all its bounded orbits are closed, therefore, periodic curves. It was shown very many years ago that there are two Bertrand potentials in Euclidean space: harmonic oscillator and attractive Coulomb problem. Incidentally, it turned out that it is the peculiarity of dimension three that the attractive Coulomb problem, i.e., inverse-square rule for the potential is also the Green function for the Laplace-Beltrami operator. A very inventive proof of the Bertrand theorem may be found, e.g., in the Arnold book "Mathematical methods of Classical Mechanics" . The natural question arises as to the non-Euclidean analogue of this theorem. One can show that there are two natural curved-space counterparts, namely ones in the constant-curvature spaces: the spherical space and the pseudo-spherical, i.e., Lobatchevski space. This may be shown in two ways; either by a kind of ths direct repeating of the Arnold proof in the Euclidean space, or in a bit more sophisticated way, by performing the projective transformation to the Euclidean Bertrand problems. By "projective" we mean such a transformation which does preserve the system of geodetic curves, but without preserving their natural parametrization. It may be shown that in the spherical space of the radius $R, SO(3, R)$ the corresponding "Coulomb" potential has the form: $V(r) = -(\alpha/R)\cot r/R$, and the corresponding "oscillator" problem is: $V(r) = (kR^2/2)\tan^2 r/R$. Obviously, here $k > 0$, but in principle it may have both possible signs. The sign decides only which pole; the "northern" one $r = 0$ or the "southern" one $r = \pi R$ is to be attractive or repulsing. The problem becomes a bit more complicated in the "elliptic space" where r is assumed to change only between 0 and $\pi R/2$. To be honest, on the spherical space there is an additional "Bertrand potential", namely geodetic one when $\alpha = 0$, and $k = 0$. The possible motions are then the great circles. In the Lobatchevski space the corresponding potentials are give by $V(r) = -(\alpha/R)\cot r/R$ for the Coulomb case and $V(r) = (kR^2/2)\tan^2 r/R$ for the attractive oscillator. And now it must be $k > 0, \alpha > 0$. There is also an additional peculiarity of the pseudo-spherical case, namely, the attractive potential has the upper bound. For energies above it the motion fails to be bounded and periodic. There is a very interesting feature of our Bertrand problems in a curved manifolds visible within the framework of the Hamilton-Jacobi theory. Namely, the energy turns out to be a sum of the geodetic terms on the manifolds and of the "usual" terms characteristic for the spherical and pseudo-spherical geometry. This opens the possibility of strange conjectures concerning some "experimental" attempts of deciding if the Universe is closed or infinite. As usual when dealing with the highly symmetric models, there is a parallelism between classical, quasi-classical and quantum theories. On the quantum level the mentioned strange features of the energy spectra are still valid.

3 Affinely Rigid Body and Oscillatory Two-Dimensional Models

The particular case of the two-dimensional physical and material space is very special in many respects. First of all the reason is that the rotation group in two dimensions is one-dimensional, therefore, commutative. This fact enables one to find some rigorous solutions, very often in analytical form expressed in terms of known special functions of mathematical physics. In any case there exists a class of models which are both realistic from the point of view of applications and rigorously solvable at least in the sense of reduction to some special functions. The quantum dependence of wave functions on the angles of the two-polar decomposition may be simply expanded in terms of the usual Fourier series. Roughly speaking, the same procedure may be used within the classical framework. In the doubly-isotropic case, including of course the geodetic one, the whole dynamics becomes reduced to the one on the manifold of diagonal matrices, i.e., to the motion of deformation invariants. There is of course some problem here, because the $GL(2, R)$-group is not semi-simple and some difficulties appear on the level of the two-dimensional volume. But one can easily solve this difficulty by assuming some volume-stabilizing potential. In any case the resulting reduced dynamics on $SL(2, R)$ is explicitly solvable in terms of special functions, at least when the interaction potential is appropriately chose, or if it is simply absent (the isochoric-geodetic problem). For example, one can solve it for the harmonic oscillator potential or for its an-harmonic correction describing also the repulsion from the singular configuration with the vanishing two-dimensional volume. It is interesting that the similar procedure works also for the higher-dimensional situations, although the co-moving angular momentum and vorticity fail to be constants of motion then. This follows from the structure of the commutation relations of the linear group. Another problem in two-dimensional affinely-rigid dynamics has to do with the analysis of the action-angle variables and the quasi-classical and quantum dynamics in certain special kind of configuration space variables. Roughly speaking, they are somehow related to the spherical variables in the four-dimensional space, although they have a slightly different nature. It is interesting that the problem becomes now reduced to that of metrically-rigid body with one additional “non-compact” degree of freedom. Performed is a detailed analysis of degeneracy (hyper-integrability). It is very interesting that on the quantum level one obtains wave function known from the theory of rigid body and the Wigner theory of representations of the groups $SO(3, R)$ and $SU(2)$.

4 Classical and Quantum Models of Affinely Rigid Body with Degenerate Dimension

We concentrate mainly on the regular situation when the dimensions of the physical and material spaces are equal. Nevertheless one can also consider something like the "affine shell problem" , when the dimension of the material space is lower than that of the physical space. Roughly speaking, this is the dynamics of homogeneously deformable coin, when the spatial dimension equals three and the material one equals two. But of course one can consider, and we do that, a general situation. From the very fundamental point of view this is the special problem of affine mappings acting between those two spaces. But of course there are many differences in comparison with the general case. We analyze them using the concepts of Grassmann and Stiefel manifolds. In a sense there are some natural generalizations of the polar and two-polar description, there are however serious differences in comparison with the regular situations. Fortunately, in a large class of problems, one can use a simplified description based on the analogues of the polar and two-polar decomposition. Using methods of group theory we can use this representation and reduce in a sense the problem to the lower-dimensional dynamics of deformation invariants. One can solve some dynamical problems for appropriately chosen potential models, including realistic ones from the elasticity point of view. The method of deriving equations of motion is based on the properties of Poisson brackets. It turns out that the same procedure applies also to the quantum case, applicable, e.g., to molecular problems. Again one uses the procedure of quantum Poisson brackets and expansion of the wave Functions in series with respect to the matrix elements of irreducible unitary representations of the spatial and material rotation groups.

5 Four-Dimensional Rigid Body and the Related Three-Dimensional Two-Gyroscopic Problems

It is well-known that there is one exceptional rotation group among all SO(n,R) with n greater than two. This is SO(4,R). Its Lie algebra is not semi-simple and this is just its particular feature. To be honest, this has also some consequences in the structure of the Lorentz group in four dimensions. It is perhaps not a very serious remark, but according to the the "antropic" principle there were some speculations concerning the particular dimension "four" of the physical space-time. But in any case it is interesting to review the dynamics of a rigid body in the four-dimensional Euclidean space. There are a few important groups related to $SO(4,R)$: the universal covering $SU(2)XSU(2)$ and some related groups like $SU(2)XSO(3,R), SO(3,R)XSU(2)$ and finally $SO(3,R)XSO(3,R)$. Of course, all of them are mutually locally isomorphic, but there is no global isomorphism, in particular $SO(4,R)$ is different than $SO(3,R)XSO(3,R)$. Their Lie algebras are identical, but on the level of the group structure there are differences, namely resulting from the division by finite groups. We show that there are a few ways to interpret the group $SO(4,R)$as a configuration space. Namely, one can consider an abstract rigid body with the $SO(4,R)XSO(4,R)$ degrees of freedom, but one can also discuss the "small" rigid body moving along the $SO(3,R)$, or better $SU(2)$ (i.e. three-dimensional sphere in the four-dimensional vector space). And finally one can consider the covering space rigid body moving along the sphere, this is again $SU(2)XSU(2)$. Some interesting problems concerning integrability and degeneracy appear then and also some interesting connection with the Bertrand problem with gyroscopic degrees of freedom. There are some rigorous formulas for the quantum energy levels of the geodetic problems and ones with the potential energy term.