Non-local and Local Nonlinear Shrödinger Equation from Geometric Curve Flows in Low Dimensional Hermitian Symmetric Space

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ABSTRACT

In recent work of [1], several nonlocal generalization of the nonlinear Shrödinger equation have been obtained together with their Lax pair, bi-Hamiltonian operators which are $U(1)$- invariant. These nonlocal Schrödiner generalization are derived from a geometrical flows utilized by a geometrical parallel moving frame on the flows in the Riemannian symmetric space $SO(2n)/U(n)$. In this set up, which is adaption of a more general result [3], the Cartan structure equation encode bi-Hamiltonian structure and Lax pair for the nonlinear nonlocal Schrödineger.

In the Hermitian symmetric spaces $SP(2)/U(2)$ and $SU(3)/U(2)$ there are a natural complex structure compatible with the Riemannian metric. The frame bundle of these spaces have subgroup $U(2)$ as a gauge group respectively. For arclength parametrized curves in this geometry, there is a natural parallel frame whose equivalence group is $U(1)$. The components of Cartan connection matrix of this frame, known as Hasimoto variables, yield a real-valued differential covariant of the curve in addition to a complex valued differential covariant. The resulting generalization of the NLS equation are $U(1)-$ invariant integrable systems in which a real variable is coupled to a complex scaler variable. We use the Hermitian structure to complexify the Hasimoto real varianble in a natural way.

The main result is utilizing two representation of singular elements in a Cartan subspaces of $\mathfrak{sp}(2)/\mathfrak{u}(2)$ and one singular element in $\mathfrak{su}(3)/\mathfrak{u}(3)$ generating center respectively in the gauge Lie subalgebras $\mathfrak{u}(2)$ and $\mathfrak{u}(3)$ of maximal dimension. The parallel frame araising in this manner in the first case yields two different new generalized non-local nonlinear Schroödinger equations for a real variable u and complex Hasimoto like variable $\mathbf{u}$ in the first case and local nonlinear Schroödinger equations in the second case. The

nonlocal one is given as

$$\mathrm{u}_t = \mathrm{u}|\mathbf{u}|^2 + Re(\bar{\mathbf{u}}_x D_x^{-1}(\mathrm{u}\mathbf{u})) \tag{1}$$

$$\mathbf{u}_t = \mathrm{i}\big(\frac{1}{4}\mathbf{u}_{xx} + \frac{1}{2}\mathbf{u}|\mathbf{u}|^2 + +\mathrm{u}^2\mathbf{u} + \mathrm{u}_x D_x^{-1}(\mathrm{u}\mathbf{u}) + 2\mathbf{u}|D_x^{-1}(\mathrm{u}\mathbf{u})|^2\big) \tag{2}$$

We also give the flow equations which is variant of Schrödinger map equation and is invariant under isometry group.

These result may also viewed as a generalization as well as a geometric interpretation of AKNS method [4] on Hermitian symmetric spaces in which they only have found local generalization of Schrödinger equation, while by construction we obtain a nonlocal one in the first case.

Toward the classification of all integrable hierarchies like generalized Sine-Gordon, local and nonlocal NLS and mKdV system by studying the geometric curve flows in all irreducible symmetric spaces, which are classified by Satake diagram, one can illustrate the whole picture and classification, by attaching to a specific vertex of each Satake diagaram all these integrable systems in each case. The present work is an example of attempt to build this correspondence in general case. The idea goes back to the different approaching the classification of KdV type equations corresponding to an arbitrary simple Lie algebras of Drinfeld and Sokolov [5] , which is actually is using normalized Lax pair in the context of Kac-Moody algebra associated to the simple Lie algebra under consideration

References

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