ERLANGEN PROGRAM IN GEOMETRY AND ANALYSIS: THE GROUP $SL_2(\mathbb{R})$ CASE STUDY

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1. Abstract

The Erlangen program of F.Klein (influenced by S.Lie) defines geometry as a study of invariants under a certain group action. This approach proved to be fruitful much beyond the traditional geometry. For example, special relativity is the study of invariants of Minkowski space-time under the Lorentz group action. Another example is complex analysis as study of objects invariant under the conformal maps.

In this course we consider in details the group $SL_2(\mathbb{R})$ and corresponding geometrical and analytical invariants with their interrelations. The course has a multi-subject nature touching algebra, geometry and analysis.

There are no prerequisites beyond a standard undergraduate curriculum: elements of group theory, linear algebra, real and complex analysis. Some knowledge of Lie groups and Hilbert spaces would be helpful but is not obligatory.

The best approximation to the Lecture Notes is the paper *Erlangen Program at Large: Outline* [6]. The geometrical part of the course (first two lectures) is covered in much details in the book [7]. The rest of the course is based on several recent papers and unpublished results.

2. Outline

Lecture 1: Structure of the group $SL_2(\mathbb{R})$ and its action on homogeneous spaces.
- The group $SL_2(\mathbb{R})$ and its Lie algebra, the Iwasawa decomposition.
- Möbius transformations of the real line.
- Subgroups of $SL_2(\mathbb{R})$ and the respective homogeneous spaces.
- Parametrisation of homogeneous spaces: complex, dual and double numbers.
- Möbius transformations in the upper half-plane.

Lecture 2: Geometry of invariants under $SL_2(\mathbb{R})$ action.
- Cycles (quadrics) as geometric SL(2,R)-invariants.
- Schwerdtfeger–Filmore–Springer–Cnops construction and algebraic invariants of cycles.
- Poincare extension of Möbius transformations of the real line.
- Algebraic and metric structures associated with "rotation" subgroups.

Lecture 3: Linear representations of the group $SL_2(\mathbb{R})$.
- Constructions of induced representation in the sense of Mackey.
- Hypercomplex characters and representations they induce.
- Unitarisation of induced representations.
- Derived representations of the Lie algebra.

Lecture 4: Theory of analytic functions in terms of representations of $SL_2(\mathbb{R})$.
- The Hardy and Bergman spaces as irreducible invariant subspaces.

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- Cauchy integral formula as a wavelet transform.
- Cauchy-Riemann and Laplace equations from the invariant vector fields.
- Laurent and Taylor expansions over the eigenvectors of rotations.
- Hypercomplex versions of analytic functions.

Lecture 5: Functional calculus and spectrum of an operator and $\mathrm{SL}_2(\mathbb{R})$ action on Banach algebras.

- Algebraic definition of the functional calculus as an algebra homomorphism.
- Covariant functional calculus as an intertwining operator.
- Prolongation of representations and functional calculus of non-selfadjoint operators.
- Spectrum of an operator as a support of the functional calculus.

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