

# Clifford's Geometric Algebra

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## Abstract

If things were as they should be, Clifford algebra would be a central feature of undergraduate mathematics. Around 1900, there was a debate amongs mathematicians and physicists between those who advocated quaternions and those who advocated vectors for the study of geometry and physics. Quaternions were superior for dealing with rotations. This debate ended in 1916 when Einstein published his general theory of relativity. It was then clear that the study of spaces with dimension higher than three was important both for geometers and physicists. Quaternions would not be sufficient for the task and vectors won.

Absent in this debate was the possibility of using Clifford's geometric algebra which includes both quaternions and vectors as subspaces. William Kingdon Clifford died in 1879 at the age of 33 and was not around to participate in the debate. Clifford was articulate both as a speaker and a writer so his geometric algebra would probably have won out had he lived another 20 or 30 years.

He wrote two papers on the subject before he died. Although Clifford was recognized as one of England's most distinguished mathematicians the two papers were overlooked and ignored. Why? The first paper appeared in the very first issue of the *American Journal of Mathematics*. Now the journal is one of the most widely read math journals in the world but few read that first issue. Why did Clifford choose an obscure journal to publish that paper? In 1876, James Joseph Sylvester was entrusted with the task of constructing the first American doctoral program in mathematics at John Hopkins University in Baltimore. Two years later he founded the *American Journal of Mathematics*. William Kingdon Clifford had a very high regard for Sylvester and presumably decided to give his support to Sylvester's endeavors.

The second paper was published after the death of Clifford in unfinished form as part of his collected papers but this too was largely overlooked.

In 1946 and 1958, Marcel Riesz published some results of Clifford algebra that stimulated David Hestenes to investigate the subject. In 1966, David Hestenes published a thin volume entitled *Space-Time Algebra*. And 18 years later, with his student Garrett Sobczyk, he wrote a more extensive work entitled *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics*. Since then, extensive research has been carried out in Clifford algebra with a multitude of applications.

During the twentieth century, the formalism of tangent vectors (directional derivatives) and differential forms has become the preferred framework for the study of Riemannian and quasi-Riemannian spaces. Perhaps there is a necessity for using this cumbersome formalism for spaces without a metric. However for spaces with a metric, the formalism of Clifford algebra is conceptually much simpler and in my opinion, more powerful.

In the following five lectures, I will try to persuade you that indeed, the formalism of Clifford algebra is useful.

## Outline

### Lecture 1. What is a Clifford algebra?

- 1.1 Definition of a Clifford algebra with a not necessarily positive definite metric.
- 1.2 Generalized reflections and rotations.
- 1.3 Coordinate bases.

### Lecture 2.

- 2.1 The mathematics of the spinning top without Euler angles.

### Lecture 3.

- 3.1 The geodesic derivative compared with the covariant derivative.
- 3.2 Moving frames and Fock-Ivanenko coefficients.

### Lecture 4.

- 4.1 Using Fock-Ivanenko coefficients to derive the Schwarzschild metric.

### Lecture 5.

- 5.1 Dirac's equation for the electron.
- 5.2 A variety of currents that can be derived from Dirac's equation.

## References

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