THE HEISENBERG GROUP AND $\mathrm{SL}_2(\mathbb{R})$: A SURVIVING PACK FOR EVERYONE

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The classification of low-dimensional (≤ 3) Lie algebras can be roughly summarised as follows. Non-commutative Lie groups start at the dimensionality 2 and the only example in this dimension is the affine group of the real line, also known as the "$ax+b$" group. The dimensionality three brought a bigger variety: there are even infinitely many non-equivalent Lie algebras. However, if we will drop various straightforward extensions of the affine group, there are essentially only two new examples: the Heisenberg group $\mathbb{H}$ and the $\mathrm{SL}_2(\mathbb{R})$ group. It is not surprising, that the fundamental origin of these two groups correlates with their importance.

To begin with, $\mathbb{H}$ initiates the class of nilpotent Lie groups and $\mathrm{SL}_2(\mathbb{R})$ is the smallest semisimple Lie group. In a sense, those two classes are bearing many opposite properties. We also shall note that $\mathrm{SL}_2(\mathbb{R})$ contains the "$ax+b$" group as a subgroup.

For a wider community, $\mathbb{H}$ and $\mathrm{SL}_2(\mathbb{R})$ are significant for their applications:

- The Heisenberg group appears in harmonic analysis, quantum mechanics, signal processing, number theory and many other areas.
- The affine group—a part of $\mathrm{SL}_2(\mathbb{R})$—is behind of numerous techniques in real analysis, wavelet theory, etc. Furthermore, the group $\mathrm{SL}_2(\mathbb{R})$ itself
  - is the group of holomorphic automorphisms of the unit disk (or the upper half-plane) in complex analysis;
  - has the Lie algebra $\mathfrak{sl}_2$ spanned by the Hamiltonian of the quantum harmonic oscillator and the respective ladder operators;
  - has the discrete subgroup $\mathrm{SL}_2(\mathbb{Z})$ consisting of the integer 2×2 matrices, which is crucial for number theory (including the celebrated proof of the Fermat Last Theorem).

The above two lists are partially overlapping and partially complementing. There are deep reasons for this. The mentioned generators of the Lie algebra $\mathfrak{sl}_2$—the Hamiltonian of the harmonic oscillator and the ladder operators—are quadratic elements of the Heisenberg Lie algebra, In the opposite direction: the group $\mathrm{SL}_2(\mathbb{R})$ naturally acts by outer automorphisms of the Heisenberg group, thus one can build the semidirect product $\mathbb{S} = \mathbb{H} \rtimes \mathrm{SL}_2(\mathbb{R})$, which is known as the Schrödinger group.

The Schrödinger group naturally covers any of the above applications of $\mathbb{H}$ and $\mathrm{SL}_2(\mathbb{R})$ and especially efficient in the areas common to both: harmonic analysis, quantum mechanics and number theory. It is enough to mention that:

- $\mathbb{S}$ is the full group of symmetries of the wave and Schrödinger equations (hence its name).
- The discrete subgroup of $\mathbb{S}$ with integer components defines the theta function.

Our course will be a gentle and non-technical introduction to this circle of ideas.

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