We will present results on matrix generalized inverses with some applications. It is well-known that a square complex matrix $A$ is invertible if and only if $\det(A) \neq 0$. In this case $AA^{-1} = A^{-1}A = I$, where $I$ is the identity matrix of the appropriate dimension. However, in many situations the considered matrix is not invertible, or not square.

There are several ways to define a generalized inverse of a general complex matrix. If the matrix $A \in \mathbb{C}^{m \times n}$ is given, then the Moore-Penrose of $A$ is the matrix $A^\dagger \in \mathbb{C}^{n \times m}$ satisfying the conditions

\begin{align*}
(1) \quad & AA^\dagger A = A, \\
(2) \quad & A^\dagger AA^\dagger = A^\dagger, \\
(3) \quad & (AA^\dagger)^* = AA^\dagger, \\
(4) \quad & (A^\dagger A)^* = A^\dagger A.
\end{align*}

The Moore-Penrose inverse of a complex matrix always exists and it is unique. If $A$ is square and invertible, then $A^\dagger = A^{-1}$, i.e. in this case the ordinary and the Moore-Penrose inverse of $A$ coincide.

Furthermore, if a vector $b \in \mathbb{C}^n$ is also given, we want to solve the equation $Ax = b$. In the most general situation, the obvious candidate for a solution is $x = A^\dagger b$. Such $x$ minimizes the norm $\|Ax - b\|$ and such $x$ has the minimum norm $\|x\|$ among all other minimizers. This is the approximation property of the Moore-Penrose inverse. The Moore-Penrose inverse is also connected with the Singular value decomposition of a complex matrix and linear regression.

Now, is $S \subset \{1, 2, 3, 4\}$, we can define a generalized inverse $A^{(S)}$ of $A$, such that only equations from $S$ are satisfied. Thus, we obtain the following classes of generalized inverses: $\{1\}$-inverses, $\{2\}$-inverses, $\{1, 2\}$-inverses, etc. This classes are also important in many applications.

If we consider a square matrix $A \in \mathbb{C}^{m \times m}$, then the commutativity conditions can be involved. For previously given $A$, the Drazin inverse $A^D \in \mathbb{C}^{m \times m}$ satisfy the conditions

\begin{align*}
(2) \quad & A^D AA^D = A^D, \\
(5) \quad & AA^D = A^D A, \quad A^{n+1} A^D = A^n
\end{align*}

for some $n \in \mathbb{N}_0$. The least such $n$ is called the Drazin index of $A$, denoted by $\text{ind}(A)$. The Drazin inverse $A^D$ of $A \in \mathbb{C}^{m \times m}$ is unique and always exists. We have $\text{ind}(A) = 0$ if and only if $A$ is invertible, and in this case $A^D = A^{-1}$. 

If \( \text{ind}(A) \leq 1 \), then the multiplicative semigroup generated by \( \{A, A^D\} \) is actually a group with the unit \( AA^D \). For this special reason, if \( \text{ind}(A) \leq 1 \), then \( A^D = A^# \) is the group inverse of \( A \). Thus, the group inverse of \( A \), \( \text{ind}(A) \leq 1 \), is the unique matrix \( A^# \) satisfying

\[
(1) \quad AA^#A = A, \quad (2) \quad A^DAA^D = A^D, \quad (5) \quad AA^D = A^DA.
\]

If \( J \) is the Jordan normal form of \( A \in \mathbb{C}^{m \times n} \), and \( J(0) \) is the appropriate Jordan block corresponding to the eigenvalue \( \{0\} \) (if \( J(0) \) exists), then \( \text{ind}(A) \) is the index of nilpotency of \( J(0) \). Of course, \( J(0) \) does not exist if and only if \( \text{ind}(A) = 0 \).

Drazin inverse have applications in finite Markov chains and singular systems of differential equations.

Lectures will be given according to the references and with the following contents.

**Lecture 1.** Moore-Penrose inverse, \( \{1\}-, \{2\}- \) and \( \{1,2\}- \) generalized inverses and related topics.

**Lecture 2.** Drazin inverse and group inverse.

**Lecture 3.** Computation of generalized inverses and solving matrix equations.

**Lecture 4 and Lecture 5.** Applications of generalized inverses: parallel sums and shorted matrices, linear regression, finite Markov chains, singular systems of differential equations.

**References**

