Quantum Groups and Stochastic Models

Stochastic reaction-diffusion processes are of both theoretical and experimental interest not only because they describe various mechanisms in physics and chemistry but they also provide a way of modelling phenomena like traffic flow, kinetics of biopolimerization, interface growth.

A stochastic process is described in terms of a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1, 2...., n-1$ at a site $i = 1, 2, ...L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s_i' during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. The rates $\Gamma \equiv \Gamma^{ik}_{jl}$ are assumed to be independent from the position in the bulk. At the boundaries, i.e. sites

1 and L additional processes can take place with rates L and R. Due to probability conservation

$$\Gamma(s,s) = -\sum_{s' \neq s} \Gamma(s',s) \tag{1}$$

The master equation for the time evolution of a stochastic system

$$\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t) \tag{2}$$

is mapped to a Schroedinger equation for a quantum Hamiltonian in imaginary time

$$\frac{dP(t)}{dt} = -HP(t) \tag{3}$$

where

$$H = \sum_{j} H_{j,j+1} + H^{(L)} + H^{(R)} \tag{4}$$

The ground state of this in general non-hermitean Hamiltonian corresponds to the stationary probability distribution of the stochastic dynamics. The mapping provides a connection with integrable quantum spin chains.

Examples - particles hop between lattice sites i, j with rates g_{ij} with a hard core repulsion (i.e. a site is emty or occupied by one particle)

1. The symmetric exclusion process - $g_{ij} = g_{ji}$. The stochastic Hamiltonian is the $SU(2)$ symmetric spin 1/2 isotropic Heisenberg ferromagnet

$$H = -\frac{1}{2}\sum_{i}(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y + \sigma_i^z\sigma_j^z - 1) \qquad (5)$$

The $SU(2)$ symmetry, yet unrevealed in the original master equation becomes manifest through the mapping and allows for exact results of the stochastic dynamics.

2. The ASEP - a diffusion driven lattice gas of particles with rates $\frac{g_{i,i+1}}{g_{i+1,i}} = q \neq 1$ is mapped to a $SU_q(2)$-symmetric XXZ chain with anizotropy $\Delta = \frac{(q+q^{-1})}{2}$.

MATRIX PRODUCT GROUND STATES APPROACH

The stationary probability distribution, i.e. the ground state of the quantum Hamiltonian is expressed as a product of (or a trace over) matrices that form representation of a quadratic algebra determined by the dynamics of the process. (Derrida et. al.- ASEP with open boundaries; 3-species diffusion-type, reaction-diffusion processes)

ANZATZ

Any zero energy eigenstate of a Hamiltonian with nearest neighbour interaction in the bulk and single site boundary terms can be written as a matrix product state with respect to a quadratic algebra

$$\Gamma^{ik}_{jl} D_i D_k = x_l D_j - x_j D_l$$

DIFFUSION - $\Gamma^{ik}_{ki} = g_{ik}$

Consider n species diffusion process on a chain with L sites with nearest-neighbour interaction with exclusion, on successive sites the particles i and k exchange places with probability g_{ik}dt; particles number n_i in the bulk is conserved

$$\sum_{i=0}^{n-1} n_i = L \tag{6}$$

Open systems with boundary processes - at site 1 (left) and at site L (right) the particle i is replaced by the particle k with probabilities $L_k^i dt$ and $R_k^i dt$ respectively.

$$L_i^i = -\sum_{j=0}^{L-1} L_j^i, \qquad R_i^i = -\sum_{j=0}^{L-1} R_j^i \tag{7}$$

DIFFUSION ALGEBRA

$$g_{ik} D_i D_k - g_{ki} D_k D_i = x_k D_i - x_i D_k \tag{8}$$

where $i, k = 0, 1, ...n-1$ and x_i are c-numbers

$$\sum_{i=0}^{n-1} x_i = 0$$

This is an algebra with INVOLUTION, hence hermitean D_i

$$D_i = D_i^{+}, \qquad g_{ik}^{+} = -g_{ki} \qquad x_i = x_i^{+} \tag{9}$$

(or $D_i = -D_i^{+}$, if $g_{ik} = g_{ki}^{+}$).

PROBABILITY DISTRIBUTION:

- periodic boundary conditions

$$P(s_1, ...s_L) = Tr(D_{s_1} D_{s_2} ... D_{s_L}) \tag{10}$$

-boundary processes

$$P(s_1, ...s_L) = < w|D_{s_1} D_{s_2} ... D_{s_L}|v > \tag{11}$$

the vectors $|v>$ and $<w|$ are defined by

$$< w|(L_i^k D_k + x_i) = 0, \qquad (R_i^k D_k - x_i)|v >= 0 \tag{12}$$

THUS to find the stationary probability distribution one has to compute traces or matrix elements with respect to the vectors $|v>$ and $<w|$ of monomials of the form

$$D_{s_1}^{m_1} D_{s_2}^{m_2} .....D_{s_L}^{m_L} \tag{13}$$

The problem to be solved is twofold - Find a representation of the matrices D that is a solution of the quadratic algebra and match the algebraic solution with the boundary conditions.

The relations (8) allow an ordering of the elements D_k. Monomials of given order are the Poincare- Birkhoff-Witt (PBW) basis for polynomials of fixed degree as the probability distribution is due to the conservation laws (6). Consider the associative algebra generated by an unit e and n elements D_k obeying $n(n-1)/2$ relations (8). The alphabetically ordered monomials

$$D_{s_1}^{n_1} D_{s_2}^{n_2} .... D_{s_l}^{n_l}, \tag{14}$$

where $s_1 < s_2 < ....s_l$, $l \geq 1$ and $n_1, n_2, ...n_l$ are non-negative integers, are a linear basis in the algebra, the PBW basis.

BRAID ASSOCIATIVITY - coincidence of two different ways of ordering which is sufficient to verify for cubic monomials only with the corresponding relations for the rates.

PROPOSITION:

1. In the case of Lie-algebra type diffusion algebras the n generators D_i, and e can be mapped to the generators J_{jk} of $GL(n)$ and the mapping is invertible. The UEA generated by D_i belongs to the UEA of the Lie-algebra of $GL(n)$.

2. The multiparameter quantized noncommutative space can be realized equivalently as a q-deformed Heisenberg algebra of n oscillators depending on $n(n-1)/2+1$ parameters (or in general on $n(n-1)/2+n$ parameters). The UEA of the elements D_i in the case of a diffusion algebra with all coefficients x_i on the RHS of eq.(8) equal to zero belongs to the UEA of a

multiparameter deformed Heisenberg algebra to which a consistent multiparameter $GL_q(n)$ quantization corresponds.

3. In an algebra with x-terms on the RHS of (8) only then is braid associativity satisfied if out of the cofficients x_i, x_k, x_l corresponding to a triple $D_i D_k D_l$ either one coefficient x is zero or two coefficients x are zero and the rates are respectively related. The diffusion algebras in this case can be obtained by either a change of basis in the n-dimensional noncommutative space or by a suitable change of basis of the lower dimensional quantum space. The appearence of the nonzero linear terms in the RHS of the quantum plane relations leads to a lower dimensional noncommutative space and a reduction of the $GL_q(n)$ invariance.

NOTE - the diffusion algebra has always the one- dimensional representations with the corresponding relations for the rates.

Representations of the diffusion algebras

A. Lie-algebra types

1. All rates equal, $g_{ij} = g_{ji} = g$

The algebra after rescaling the generators $D_i, i = 0, 1, 2, ...n-1$ by

$$D_i = \frac{x_i}{g} D'_i, \qquad \sum_{i=1}^{n-1} x_i = 0 \qquad (15)$$

takes the form

$$\begin{aligned} [D_0, D_1] &= D_0 - D_1 \\ [D_0, D_2] &= D_0 - D_2 \\ &\vdots \\ [D_{n-2}, D_{n-1}] &= D_{n-2} - D_{n-1} \end{aligned} \qquad (16)$$

These algebraic relations are solved in terms of the $GL(n)$ Lie-algebra generators J_i^j:

$$\begin{aligned} D_0 &= J_0^0 + J_0^1 + J_0^2 + ... + J_0^{n-1} \\ D_1 &= J_1^0 + J_1^1 + J_1^2 + ... + J_1^{n-1} \\ D_2 &= J_2^0 + J_2^1 + J_2^2 + ... + J_2^{n-1} \\ &\vdots \\ D_{n-1} &= J_{n-1}^0 + J_{n-1}^1 + J_{n-1}^2 + ... J_{n-1}^{n-1} \end{aligned} \qquad (17)$$

The conventional basis for fundamental representation of the $GL(n)$ generators given by the $(e_{ij})_{ab} = \delta_{ia}\delta_{jb},\ i,j,a,b = 0,1,2...n-1$ provides the n-dimensional matrix representation of the generators D, with entries 1 in only the first row of D_0, the second row of D_1, the third row of D_3,...the last row of D_{n-1} and all the entries elsewhere zero. The correspondence is one-to-one since

$$J_i^j = \frac{1}{n} D_i D_j^T \tag{18}$$

The Poincare-Birkhoff-Witt basis of the algebra generated by the elements D is a subsystem of the basis of the universal enveloping algebra of $sl(n) \oplus u(1)$ which is the hidden symmetry algebra of a stochastic diffusion system with all rates equal.

1.1. Algebra and Boundary Problem for $n = 2$ and $n = 3$

The algebra $[D_0, D_1] = D_0 - D_1$, is solved by
$D_0 = J_0^0 + J_0^1 \qquad D_1 = J_1^0 + J_1^1$
The boundary vectors are determined by the conditions

$$< w|(L_1^0 D_0 - L_0^1 D_1 + x_1) = 0 \tag{19}$$

$$(-R_1^0 D_0 + R_0^1 D_1 - x_0)|v >= 0 \tag{20}$$

with $x_0 + x_1 = 0$. The boundary matrices are simultaneously diagonalized with the constraints

$$L_1^0 + L_0^1 = g, \qquad R_0^1 + R_1^0 = -g, \tag{21}$$

CONTRADICTION - all the rates are probability rates and have to be POSITIVE. There is an algebraic solution consistent with the boundary conditions, namely

$$\begin{aligned} D_0 &= \frac{x_0}{g}((1+\alpha)J_0^0 + J_0^1 + \alpha J_1^1) \qquad (22) \\ D_1 &= \frac{x_1}{g}(\alpha J_0^0 + J_1^0 + (1+\alpha)J_1^1) \end{aligned}$$

It introduces an additional arbitrary parameter and this is the price to be paid to match the algebra with the boundary vectors which hence determines a Fock representation of the diffusion algebra with a constraint for the rates

$$g(L_0^1+L_1^0+R_0^1+R_1^0) = (L_0^1+L_1^0)(R_0^1+R_1^0) \quad (23)$$

Unlike the $n = 2$ problem the expressions for the $n = 3$ D-matrices

$$\begin{aligned} D_0 &= \frac{x_0}{g}(J_0^0 + J_0^1 + J_0^2) \quad (24) \\ D_1 &= \frac{x_1}{g}(J_1^0 + J_1^1 + J_1^2) \\ D_2 &= \frac{x_2}{g}(J_2^0 + J_2^1 + J_2^2) \end{aligned}$$

that solve the diffusion algebra yield a consistent solution for the boundary vectors. The latter are in this case determined by the systems

$$< w((-L_1^0 - L_2^0)D_0 + L_0^1 D_1 + L_0^2 D_2 + x_0)) = 0$$

$$< w(L_1^0 D_0 + (-L_0^1 - L_2^1)D_1 + L_1^2 D_2 + x_1) = 0$$

$$< w(L_2^0 D_0 + L_2^1 D_1 + (-L_0^2 - L_1^2)D_2 + x_2) = 0$$

and

$$(-R_1^0 - R_2^0)D_0 + R_0^1 D_1 + R_0^2 D_2 - x_0)v >= 0$$

$$(R_1^0 D_0 + (-R_0^1 - R_2^1) D_1 + R_1^2 D_2 - x_1) v >= 0$$

$$(R_2^0 D_0 + R_2^1 D_1 + (-R_0^2 - R_1^2) D_2 - x_2) v >= 0$$

with $x_0 + x_1 + x_2 = 0$ The parameters x provide a matching condition for a common eigenvalue zero of the left and right transition matrices with the corresponding left and right boundary vectors and constraints on the boundary rates

$$\begin{aligned} R_0^1 L_0^2 + L_0^1 R_0^2 + (L_1^0 &+ L_2^0)(R_0^1 + R_0^2) + \\ (R_1^0 + R_2^0)(L_0^1 + L_0^2) &= g(L_0^1 - L_0^2 + R_0^1 - R_0^2) \\ (R_0^1 + R_2^1) L_1^2 - (L_0^1 + L_2^1) R_1^2 &+ R_1^0 (L_0^1 + L_2^1 + L_1^2) - \\ L_1^0 (R_1^2 + R_0^1 + R_2^1) &= g(L_1^0 - L_1^2 + R_1^0 - L_1^2) \end{aligned}$$

The generalisation of these representations to general n is straightforward.
A realisation

$$J_{ik} = A_i^+ A_k \tag{25}$$

yields a representation of the elements D and the the boundary vectors in the oscillator basis.