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The Camassa-Holm Equation: Evolution of the Scattering Coefficients

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OUTLINE

- Background
- Difficulties with the CH - IST
- How to Resolve the Difficulties
- Composite IST
- CH Stationary Points
- Evolution of CH Stationary Points
- Connecting Together a Composite IST
- Evolution of the Scattering Coefficients
- Summary

BACKGROUND

Camassa and Holm; PRL **71**, 1661 (1993).

$$m_t + um_x + 2u_x(m+\kappa) = 0, \quad t>0,\ x\in R\,,$$

$$m = u - u_{xx}\,,$$

Lax Pair:

$$v_{xx} - \frac{1}{4}v + \boxed{\frac{m+\kappa}{2\lambda}}v = 0\,,$$

$$v_t + uv_x + \lambda v_x + (\alpha - \frac{1}{2}u_x)v = 0\,,$$

λ – spectral parameter

α – arbitrary constant

Features of CH

- Model for Water Waves
- The $(m + \kappa) > 0$ IST problem is done (Lenells, Constantin)
- Regular Solitons and Peaked Solitons (Boyd-2005)
- Multi-Soliton Solutions (A. Parker – 2005)

Potential

$$v_{xx} - \left(\frac{1}{4} - \frac{m+\kappa}{2\lambda} \right) v = 0$$

Spectral parameter

- Lax Eigenvalue Problem - Quite Different!
- Product of Potential and $1/\lambda$, instead of Sum

Difficulty of CH Lax Pair

If m + κ ever crosses 0 and λ approaches 0, then the asymptotics become non-uniform in *x*!

$$\text{as } \lambda \to 0\,; \quad v \to \exp\left(\pm i \int^x dx \sqrt{\frac{m+\kappa}{2\lambda}} \right)$$

1. the analytical properties of the Jost functions rotate by 90°
2. they become non-analytic after the first zero
3. without uniform asymptotics, how can we create the IST?
4. linear dispersion relations (RH) require uniformity

Other Similar Problems

Oscillating Two Stream Instability (Kaup '80):

$$\partial_\tau q_0 = -2\,\mathrm{i}\, q_0^* q_+\, q_-$$
$$\partial_\chi q_+ = -\mathrm{i}\, q_0^2\, q_-^*$$
$$\partial_\chi q_- = \mathrm{i}\, q_0^2\, q_+^*$$

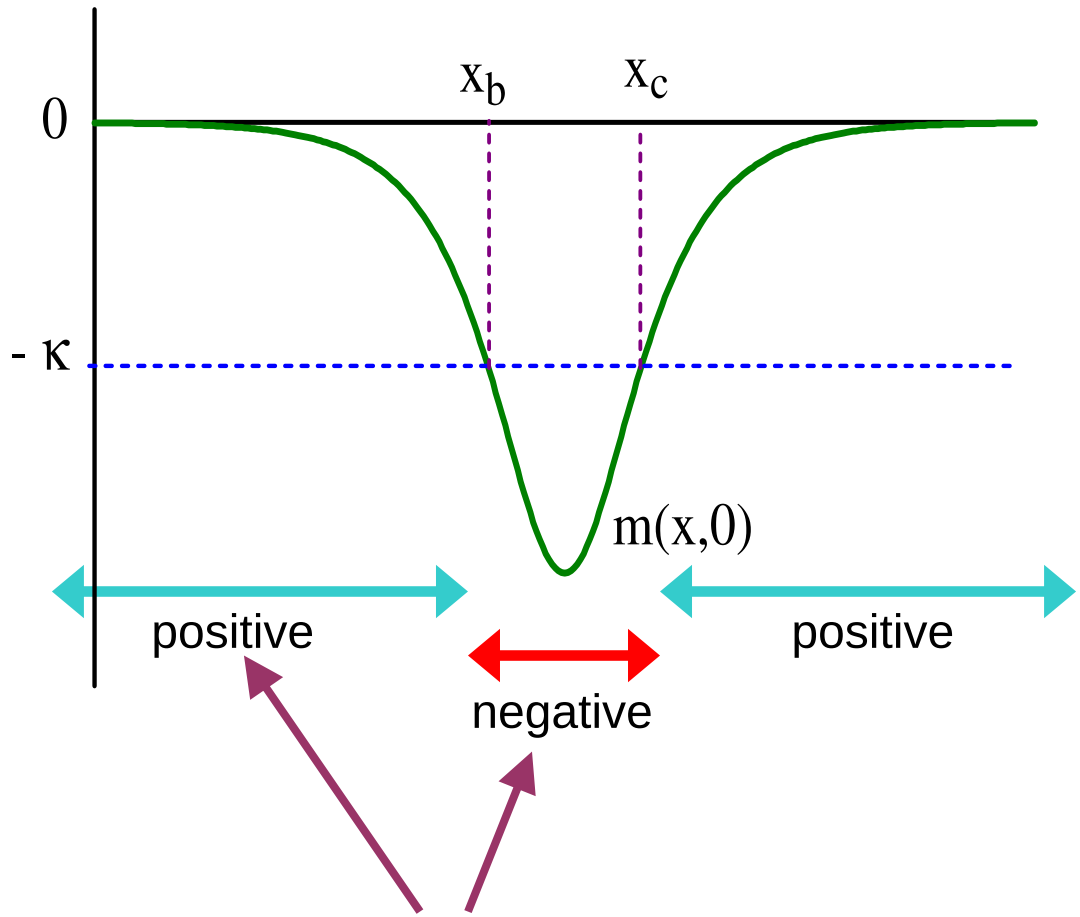
(is also semi-stable, like embedded solitons)

Degenerate Two-Photon Propagation (Kaup, Steudel '96):

$$\partial_\tau r_+ = i(S_+\, r_3 + g_1\, S_3\, r_+)$$
$$\partial_\tau r_3 = \frac{i}{2}(r_+\, S_- + r_-\, S_+)$$
$$\partial_\chi S_+ = i(-r_+\, S_3 + g_2\, r_3\, S_+)$$
$$\partial_\chi S_3 = \frac{i}{2}(r_-\, S_+ + r_+\, S_-)$$
$$S_+\, S_- - S_3^2 = 0$$

Each system above has a Lax Pair where λ multiplies the potential.

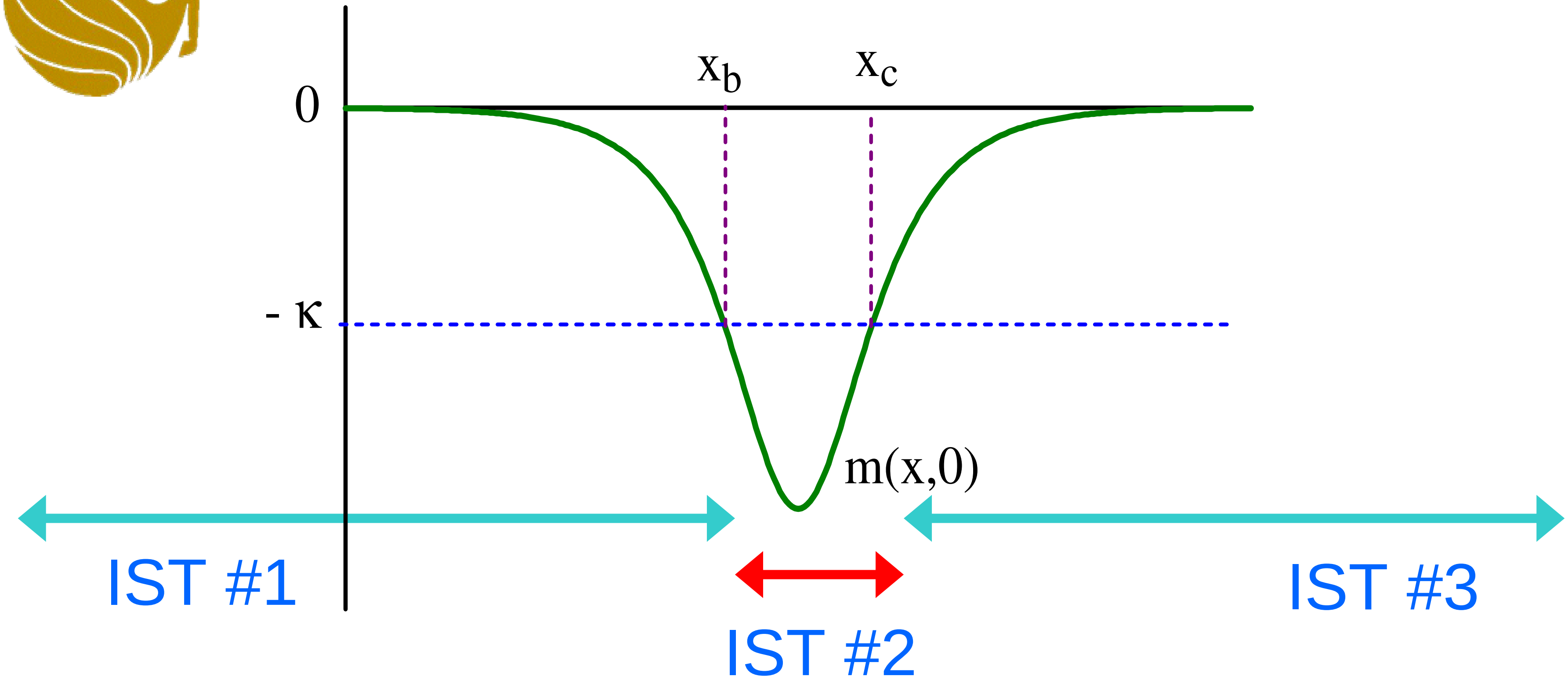
Example Initial Data for m(x,0)



Jost functions in these regions have different asymptotics:

McKean (99,04) – When m + κ is positive at the left of a negative region, as above, the solution will break in a finite time.
Will that show up in our IST?

A Solution to the Difficulty



- Each interval is either finite or semi-infinite
- In each interval – each Jost function – uniform asymptotics
- Three sets of scattering data
- Can three sets of scattering data be coupled properly? (Yes)

CH Stationary Points

The points where $(m + \kappa) = 0$, $x_{0i}(t)$, move with the flow.

$$(\partial_t + u\partial_x)(m+\kappa) + 2(m+\kappa)\partial_x u = 0$$

These are the points that define the intervals.

Thus one can define new coordinates:

$$\partial_x \chi = \sqrt{|m+\kappa|}, \quad \tau = t$$

where the endpoints of χ are invariants.

Thus the motion inside any interval, is analogous
to χ being an elastic string, but fixed at the end points.

Flow at Stationary Points

Let us expand *u(x,t)* as:

$$u(x,t) = \eta_0 + (x - x_0)\eta_1 + \frac{1}{2!}(x - x_0)^2\eta_2 + \ldots$$

Then the CH evolution gives:

$$\begin{aligned}
\eta_2 &= \eta_0 + \kappa \\
\partial_t x_0 &= \eta_0 \\
\partial_t(\eta_1 - \eta_3) + 3\eta_1(\eta_1 - \eta_3) &= 0 \\
\partial_t(\eta_2 - \eta_4) + 4\eta_1(\eta_2 - \eta_4) &= -5\eta_2(\eta_1 - \eta_3) \\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

Thus given η_0 and η_1, we have the motion of the stationary point, as well as all other η's.

Direct Scattering Problems

Will model each Jost system, in each interval, after plane waves:

$$\text{as } x \to \pm\infty \text{ and } v \to e^{\pm ikx}, \text{ then } \lambda = \frac{2\kappa}{4k^2+1}$$

In each interval, we define a ϕ and a ψ solution, satisfying:

On left:

$$\phi(x_\ell, t; k) = e^{-ikx_\ell}$$

$$\partial_x \phi(x,t;k)\big|_{x_\ell} = -i\,k\,e^{-ikx_\ell}$$

$$\overline{\phi}(x,t;k) = \phi(x,t;k^*)^*$$

On right:

$$\psi(x_r, t; k) = e^{ikx_r}$$

$$\partial_x \psi(x,t;k)\big|_{x_r} = i\,k\,e^{ikx_r}$$

$$\overline{\psi}(x,t;k) = \psi(x,t;k^*)^*$$

Then we take:

$$\begin{pmatrix} \phi \\ \overline{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} a(k,t) & b(k,t) \\ \overline{b}(k,t) & \overline{a}(k,t) \end{pmatrix}}_{\text{Scattering Coefficients}} \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix}$$

Analytical Properties

In the complex k – plane:

In interval #1 (left): $\phi_1 e^{ikx}\,,\ \psi_1 e^{-ikx}\,,\ a_1 \text{ and } \underline{b_1}$ -- analytic in UHP

In interval #2 (middle): $\phi_2 e^{ikx}\,,\ \psi_2 e^{-ikx}\,,\ a_2 \text{ and } b_2$ -- entire functions

In interval #3 (right): $\phi_3 e^{ikx}\,,\ \psi_3 e^{-ikx}\,,\ a_3 \text{ and } \underline{\bar{b}_3}$ -- analytic in UHP

In all intervals:

$$\phi_j e^{ikx}\,,\ \psi_j e^{-ikx} \text{ and } a_j \to 1 \ \text{ as } |k| \to \infty$$ -- in UHP (at fixed *x*)

Similar for the conjugate quantities.

This will have important consequences later.

Time Evolution of Scattering Coefficients

This equation is now satisfied. → $$v_{xx} - \frac{1}{4}v + \frac{m+\kappa}{2\lambda}v = 0\,,$$

Lax evolution operator: $$v_t + \underline{u}v_x + \lambda v_x + (\underline{\alpha} - \frac{1}{2}\underline{u_x})v = 0\,,$$

1. Determine α from ϕ Jost function at left end.
2. Express ϕ in terms of scattering coefficients and ψ's.
3. Drop result into evolution operator at right end.
4. The flow at x_b determines the evolution.

$$\partial_t a_1 = i\eta_{0b}\frac{\kappa}{2k\lambda}a_1 + \frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b})b_1 e^{2ikx_b}$$

$$\partial_t b_1 = \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b})a_1 e^{-2ikx_b} + \frac{i}{2k}\left[\lambda - 2\kappa - \frac{\eta_{0b}\kappa}{\lambda}\right]b_1$$

Evolutions of a's and b's

$$\partial_t a_1 = i\eta_{0b}\frac{\kappa}{2k\lambda}\,a_1 + \frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b})b_1 e^{2ikx_b}$$

$$\partial_t b_1 = \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b})a_1 e^{-2ikx_b} + \frac{i}{2k}\left[\boxed{\lambda} - 2\kappa - \frac{\eta_{0b}\kappa}{\lambda}\right] b_1$$

$$\partial_t a_2 = i(\eta_{0c} - \eta_{0b})\frac{\kappa}{2k\lambda}\,a_2 + \frac{2\eta_{1c}k + i\eta_{0c}}{4k}e^{2ikx_b}b_2 - \frac{2\eta_{1b}k - i\eta_{0b}}{4k}e^{-2ikx_b}\bar{b}_2$$

$$\partial_t b_2 = \frac{1}{2}(\eta_{1c} - \frac{i}{2k}\eta_{0c})a_2 e^{-2ikx_c} - \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b})\bar{a}_2 e^{-2ikx_b} + \frac{i}{2k}\left[\boxed{\lambda} - 2\kappa - \kappa\frac{\eta_{0b} + \eta_{0c}}{\lambda}\right] b_2$$

$$\partial_t \bar{a}_2 = -i(\eta_{0c} - \eta_{0b})\frac{\kappa}{2k\lambda}\,\bar{a}_2 - \frac{2\eta_{1b}k + i\eta_{0b}}{4k}e^{2ikx_b}b_2 + \frac{2\eta_{1c}k - i\eta_{0c}}{4k}e^{-2ikx_c}\bar{b}_2$$

$$\partial_t \bar{b}_2 = -\frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b})a_2 e^{2ikx_b} + \frac{1}{2}(\eta_{1c} + \frac{i}{2k}\eta_{0c})\bar{a}_2 e^{2ikx_c} - \frac{i}{2k}\left[\boxed{\lambda} - 2\kappa - \kappa\frac{\eta_{0b} + \eta_{0c}}{\lambda}\right] \bar{b}_2$$

$$\partial_t a_3 = -i\eta_{0c}\frac{\kappa}{2k\lambda}\,a_3 - \frac{1}{2}(\eta_{1c} - \frac{i}{2k}\eta_{0c})\bar{b}_3 e^{-2ikx_c}$$

$$\partial_t \bar{b}_3 = -\frac{1}{2}(\eta_{1c} + \frac{i}{2k}\eta_{0c})a_3 e^{2ikx_c} - \frac{i}{2k}\left[\boxed{\lambda} - 2\kappa - \frac{\eta_{0c}\kappa}{\lambda}\right] \bar{b}_3$$

$$\lambda = \frac{2\kappa}{4k^2 + 1}$$

λ has poles

Analytical Properties

Solution depends on η_0 and η_1 at stationary points.

These have been unspecified and arbitrary up to now.

General solution of b's will have essential singularities in UHP and LHP.

All scattering coefficients are analytic in either UHP or LHP, or both.

Therefore, η_0 and η_1 must evolve such that no essential singularities will ever appear, as long as *m* does not "break".

Requiring all b(λ)'s to be regular, when they should, in the limit of λ approaching infinity, uniquely determines all η_0's and η_1's.

$$I_{ij}^{(\gamma)} = \int_{x_i}^{x_j} m(x,t) e^{\gamma x} dx \,, \quad \gamma = \pm 1 \text{ or } 0 \,, \quad x_a = -\infty \,, \quad x_d = +\infty$$

Near the Singular Points

k = + i/2

$$a_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad b_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$a_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad b_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$\bar{a}_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad \bar{b}_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$a_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad \bar{b}_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2})\,,$$

k = - i/2

$$\bar{a}_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad \bar{b}_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$a_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad b_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$\bar{a}_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad \bar{b}_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2})\,,$$
$$\bar{a}_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2})\,, \quad b_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2})\,.$$

Conditions and Solution:

$$\eta_{1b} - \eta_{0b} + I_{ab}^{(+1)} e^{-x_b} = 0$$

$$(\eta_{1c} - \eta_{0c})e^{x_c} - (\eta_{1b} - \eta_{0b})e^{x_b} + I_{bc}^{(+1)} = 0$$

$$(\eta_{1c} + \eta_{0c})e^{-x_c} - (\eta_{1b} + \eta_{0b})e^{-x_b} + I_{bc}^{(-1)} = 0$$

$$\eta_{1c} + \eta_{0c} - I_{cd}^{(-1)} e^{x_c} = 0$$

$$\eta_{0b} = \frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)}$$

$$\eta_{1b} = -\frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)}$$

$$\eta_{0c} = \frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)}$$

$$\eta_{1c} = -\frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)}$$

$$I_{ij}^{(\gamma)} = \int_{x_i}^{x_j} m(x,t) e^{\gamma x} dx\,, \quad \gamma = \pm 1 \text{ or } 0\,, \quad x_a = -\infty\,, \quad x_d = +\infty$$

SUMMARY

- Non-uniform asymptotics require multiple IST's.
- Also applies to the OTSI and DTPP problems.
- Location of stationary points of flow determine the intervals.
- Set up a universal set of Jost functions and scattering coefficients. (same for all intervals)
- Evolution of scattering coefficients found.
- The η's are determined by derivative of scattering coefficients at k = +- i/2.
- The η's are also found to be related to simple integrals over *m(x,t).* (dynamics?)
- Much more remains to be done.