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The Camassa-Holm Equation: Evolution of the Scattering Coefficients

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OUTLINE

- Background
- Difficulties with the CH - IST
- How to Resolve the Difficulties
- Composite IST
- CH Stationary Points
- Evolution of CH Stationary Points
- Connecting Together a Composite IST
- Evolution of the Scattering Coefficients
- Summary

BACKGROUND

Camassa and Holm; PRL **71**, 1661 (1993).

$$m_t + um_x + 2u_x(m+\kappa) = 0, \quad t > 0,\ x \in R\,,$$

$$m = u - u_{xx}\,,$$

Lax Pair:

$$v_{xx} - \frac{1}{4}v + \boxed{\frac{m+\kappa}{2\lambda}}v = 0\,,$$

$$v_t + uv_x + \lambda v_x + (\alpha - \frac{1}{2}u_x)v = 0\,,$$

λ – spectral parameter

α – arbitrary constant

Features of CH

- Model for Water Waves
- The $(m + \kappa) > 0$ IST problem is done (Lenells, Constantin)
- Regular Solitons and Peaked Solitons (Boyd-2005)
- Multi-Soliton Solutions (A. Parker – 2005)

Potential

$$v_{xx} - \left(\frac{1}{4} - \frac{m+\kappa}{2\lambda} \right) v = 0$$

Spectral parameter

- Lax Eigenvalue Problem - Quite Different!
- Product of Potential and $1/\lambda$, instead of Sum

Difficulty of CH Lax Pair

If m + κ ever crosses 0 and λ approaches 0, then the asymptotics become non-uniform in *x*!

$$\text{as } \lambda \to 0\,; \quad v \to \exp\left(\pm i \int^x dx \sqrt{\frac{m+\kappa}{2\lambda}} \right)$$

1. the analytical properties of the Jost functions rotate by 90°
2. they become non-analytic after the first zero
3. without uniform asymptotics, how can we create the IST?
4. linear dispersion relations (RH) require uniformity

Other Similar Problems

Oscillating Two Stream Instability (Kaup ‘80):

$$\partial_\tau q_0 = -2\,\mathrm{i}\, q_0^* q_+\, q_-$$
$$\partial_\chi q_+ = -\mathrm{i}\, q_0^2\, q_-^*$$
$$\partial_\chi q_- = \mathrm{i}\, q_0^2\, q_+^*$$

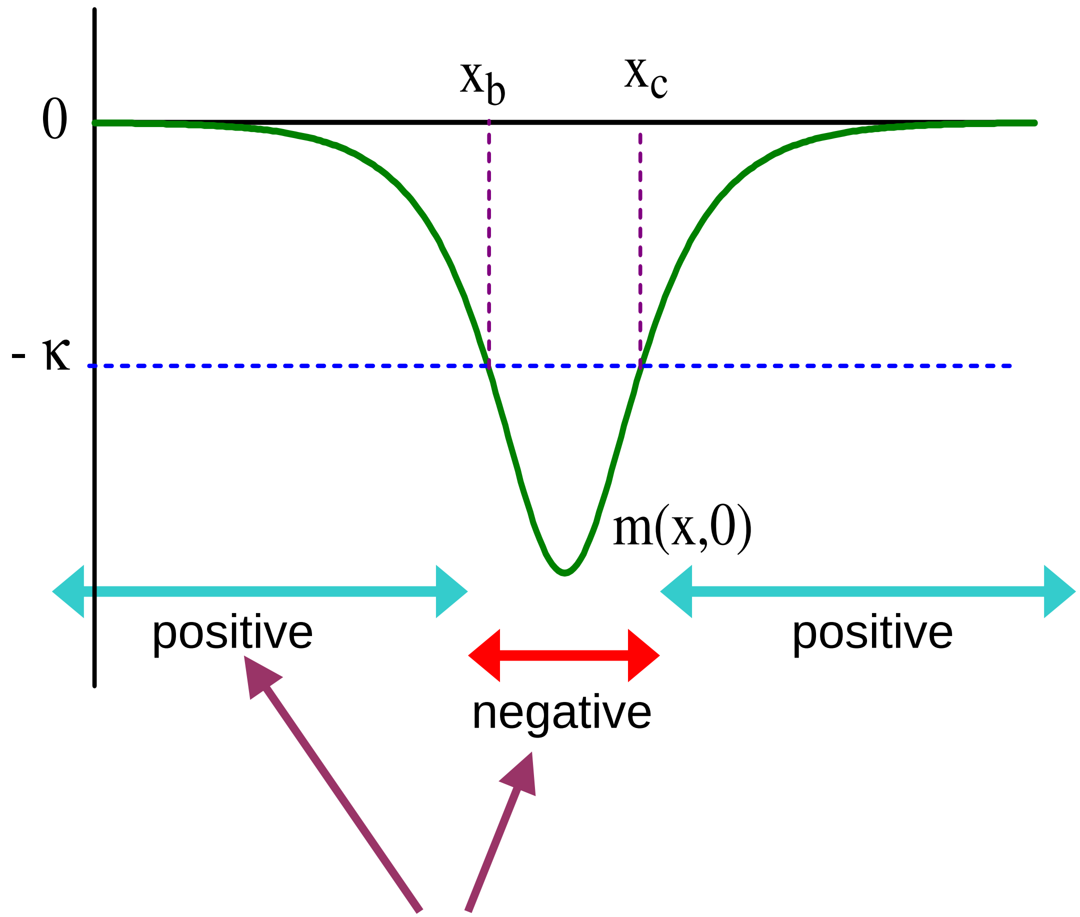
(is also semi-stable, like embedded solitons)

Degenerate Two-Photon Propagation (Kaup, Steudel ’96):

$$\partial_\tau r_+ = i(S_+\, r_3 + g_1\, S_3\, r_+)$$
$$\partial_\tau r_3 = \frac{i}{2}(r_+\, S_- + r_-\, S_+)$$
$$\partial_\chi S_+ = i(-r_+\, S_3 + g_2\, r_3\, S_+)$$
$$\partial_\chi S_3 = \frac{i}{2}(r_-\, S_+ + r_+\, S_-)$$
$$S_+\, S_- - S_3^2 = 0$$

Each system above has a Lax Pair where λ multiplies the potential.

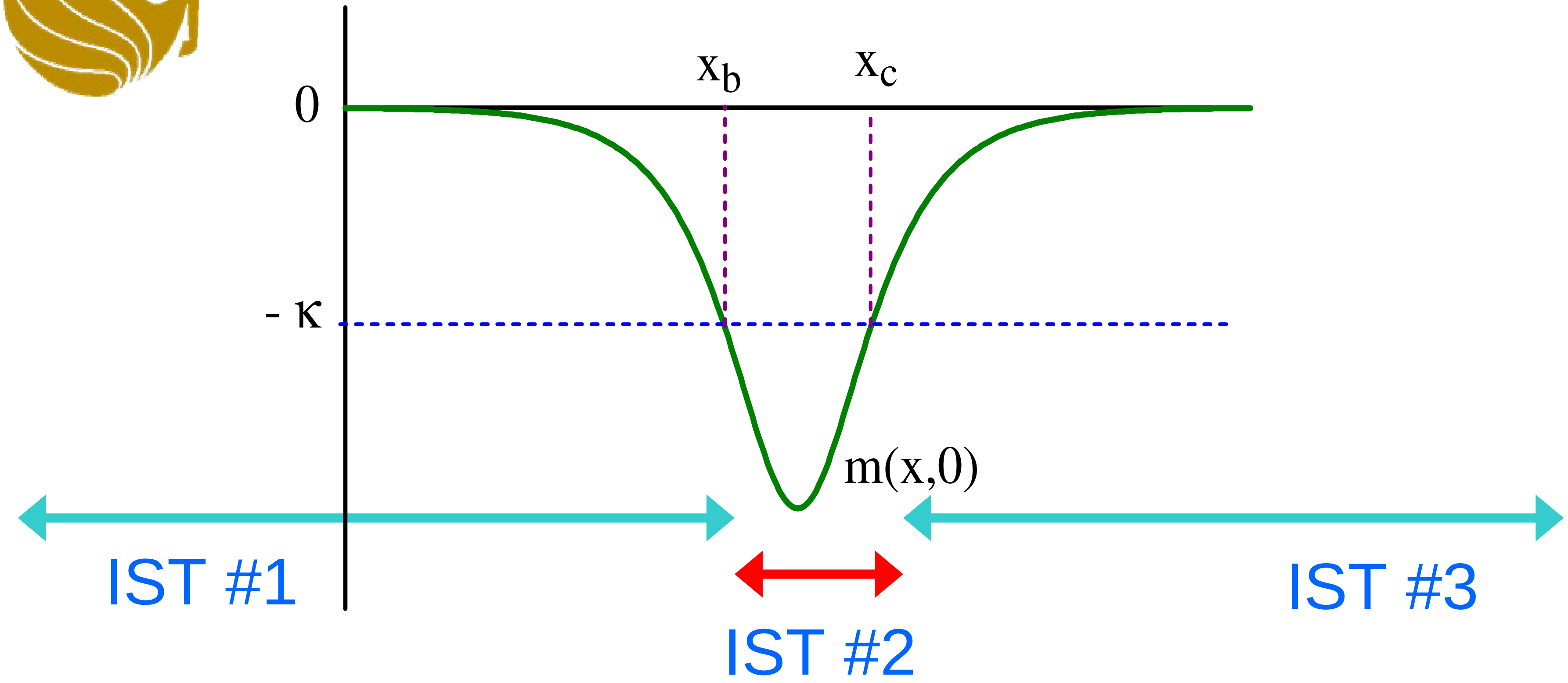
Example Initial Data for m(x,0)



Jost functions in these regions have different asymptotics:

McKean (99,04) – When m + κ is positive at the left of a negative region, as above, the solution will break in a finite time.
Will that show up in our IST?

A Solution to the Difficulty



- Each interval is either finite or semi-infinite
- In each interval – each Jost function – uniform asymptotics
- Three sets of scattering data
- Can three sets of scattering data be coupled properly? (Yes)

CH Stationary Points

The points where $(m + \kappa) = 0$, $x_{0i}(t)$, move with the flow.

$$(\partial_t + u\partial_x)(m+\kappa) + 2(m+\kappa)\partial_x u = 0$$

These are the points that define the intervals.

Thus one can define new coordinates:

$$\partial_x \chi = \sqrt{|m+\kappa|}, \quad \tau = t$$

where the endpoints of χ are invariants.

Thus the motion inside any interval, is analogous
to χ being an elastic string, but fixed at the end points.

Flow at Stationary Points

Let us expand *u(x,t)* as:

$$u(x,t) = \eta_0 + (x - x_0)\eta_1 + \frac{1}{2!}(x - x_0)^2\eta_2 + \ldots$$

Then the CH evolution gives:

$$\eta_2 = \eta_0 + \kappa$$
$$\partial_t x_0 = \eta_0$$
$$\partial_t(\eta_1 - \eta_3) + 3\eta_1(\eta_1 - \eta_3) = 0$$
$$\partial_t(\eta_2 - \eta_4) + 4\eta_1(\eta_2 - \eta_4) = -5\eta_2(\eta_1 - \eta_3)$$
$$\vdots \qquad\qquad \vdots$$

Thus given η_0 and η_1, we have the motion of the stationary point, as well as all other η's.