Hodge * and Gaussian mapping

- Hodge *

Basic properties:

$$*f = f\omega_1 \wedge \omega_2;$$

$$*\omega_1 = \omega_2, *\omega_2 = -\omega_1;$$

$$d * df = \nabla^2 f\omega_1 \wedge \omega_2$$

The second Green identity:

$$\int_M (f d * dh - h d * df) = \int_{\partial M} (f * dh - h * df)$$

[Westenholz *Differential Forms in Mathematical Physics*]

- Gaussian mapping

Gaussian mapping $\mathcal{G}: M \to S^2; \mathcal{G}(\mathbf{r}) = \mathbf{e}_3(\mathbf{r})$

Induced mapping $\mathcal{G}^\star: \Lambda^1 \to \Lambda^1$

$$\mathcal{G}^\star \omega_1 = \omega_{13},\ \mathcal{G}^\star \omega_2 = \omega_{23}$$

Define new differential operator $\tilde{d} = \mathcal{G}^\star d$

Define $\tilde{*}$: $\tilde{*}\omega_{13} = \omega_{23}, \tilde{*}\omega_{23} = -\omega_{13}$

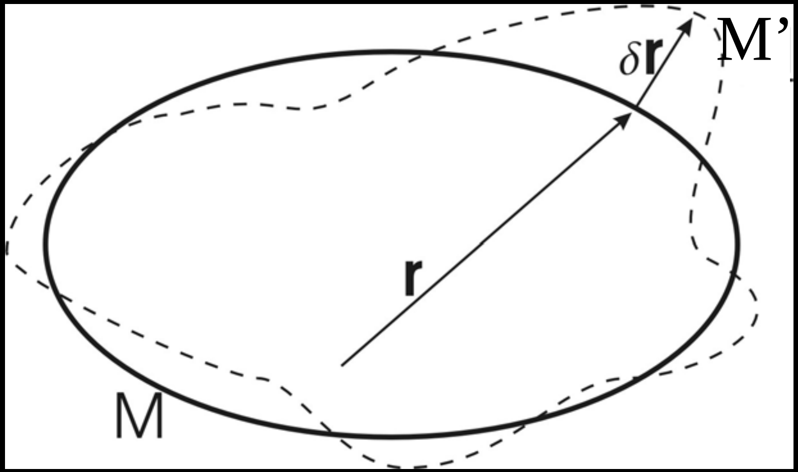
Lemma: $\int_M (f d\tilde{*}\tilde{d}h - h d\tilde{*}\tilde{d}f) = \int_{\partial M} (f\tilde{*}\tilde{d}h - h\tilde{*}df)$

Define $\nabla \cdot \tilde{\nabla}$: $d\tilde{*}\tilde{d}f = \nabla \cdot \tilde{\nabla} f \omega_1 \wedge \omega_2$

Variational theory of surface

- What is surface variation?

Each point undergoes an infinitesimal displacement



$$\delta \mathbf{r} = \delta_1 \mathbf{r} + \delta_2 \mathbf{r} + \delta_3 \mathbf{r}$$

$$\delta_i \mathbf{r} = \Omega_i \mathbf{e}_i \quad (i = 1, 2, 3)$$

Not use Einstein summation convention

- Variation of general function on a surface

If f is a generalized function of $\mathbf{r}$ (including scalar function, vector function, and r-form dependent on point $\mathbf{r}$), define

$$\delta_i^{(q)} f = (q!)\mathcal{L}^{(q)}[f(\mathbf{r} + \delta_i \mathbf{r}) - f(\mathbf{r})] \quad (i = 1, 2, 3; q = 1, 2, 3, \cdots)$$

q-order variation of f

$$\delta^{(q)} f = (q!)\mathcal{L}^{(q)}[f(\mathbf{r} + \delta \mathbf{r}) - f(\mathbf{r})] \quad (q = 1, 2, 3, \cdots)$$

$\mathcal{L}^{(q)}[\cdots]$: $\Omega_1^{q_1}\Omega_2^{q_2}\Omega_3^{q_3}$ in Taylor series
$q_1 + q_2 + q_3 = q$
q_1, q_2, q_3 being non-negative integers.

Basic properties

(i) $\delta_i^{(q)}$ and $\delta^{(q)}$ $(i = 1, 2, 3; q = 1, 2, \cdots)$ are linear operators;

(ii) $\delta_1^{(1)}$, $\delta_2^{(1)}$, $\delta_3^{(1)}$ and $\delta^{(1)}$ are commutative with each other;

(iii) $\delta_i^{(q+1)} = \delta_i^{(1)}\delta_i^{(q)}$ and $\delta^{(q+1)} = \delta^{(1)}\delta^{(q)}$,
thus we can safely replace $\delta_i^{(1)}$, $\delta_i^{(q)}$, $\delta^{(1)}$, and $\delta^{(q)}$ by
δ_i, δ_i^q, δ, and δ^q $(q = 2, 3, \cdots)$, respectively;

(iv) For functions f and g,
$\delta_i[f(\mathbf{r}) \circ g(\mathbf{r})] = \delta_i f(\mathbf{r}) \circ g(\mathbf{r}) + f(\mathbf{r}) \circ \delta_i g(\mathbf{r})$,
where $\circ$ represents the ordinary production,
vector production or exterior production;

(v) $\delta_i f[g(\mathbf{r})] = (\partial f / \partial g)\delta_i g$; (vi) $\delta^q = (\delta_1 + \delta_2 + \delta_3)^q$.

- Variational equation of frame

$$\delta_l \mathbf{e}_i = \Omega_{lij} \mathbf{e}_j, \Omega_{lij} = -\Omega_{lji} \qquad d\delta_l = \delta_l d$$

$$\begin{aligned} \delta_1 \omega_1 &= d\Omega_1 - \omega_2 \Omega_{121}, \\ \delta_1 \omega_2 &= \Omega_1 \omega_{12} - \omega_1 \Omega_{112}, \\ \Omega_{113} &= a\Omega_1, \quad \Omega_{123} = b\Omega_1 \end{aligned}$$

$$\begin{aligned} \delta_2 \omega_1 &= \Omega_2 \omega_{21} - \omega_2 \Omega_{221}, \\ \delta_2 \omega_2 &= d\Omega_2 - \omega_1 \Omega_{212}, \\ \Omega_{213} &= b\Omega_2, \quad \Omega_{223} = c\Omega_2 \end{aligned}$$

$$\begin{aligned} \delta_3 \omega_1 &= \Omega_3 \omega_{31} - \omega_2 \Omega_{321}, \\ \delta_3 \omega_2 &= \Omega_3 \omega_{32} - \omega_1 \Omega_{312}, \\ d\Omega_3 &= \Omega_{313} \omega_1 + \Omega_{323} \omega_2; \end{aligned}$$

$$\delta_l \omega_{ij} = d\Omega_{lij} + \Omega_{lik} \omega_{kj} - \omega_{ik} \Omega_{lkj}$$

Variational problem on closed surface

- Functional

$$\mathcal{F} = \int_M \mathcal{E}(2H[\mathbf{r}], K[\mathbf{r}])dA + \Delta p \int_V dV$$

$$\delta\mathcal{F} = \delta_1\mathcal{F} + \delta_2\mathcal{F} + \delta_3\mathcal{F}$$

- Lemmas

$$\delta_3 dA = -(2H)\Omega_3 dA \qquad \delta_3 \int_V dV = \int_M \Omega_3 dA$$

$$\delta_3(2H)dA = 2(2H^2 - K)\Omega_3 dA + d * d\Omega_3$$

$$\delta_3 K dA = 2KH\Omega_3 dA + d\tilde{*}\tilde{d}\Omega_3 \qquad \delta_1\mathcal{F} = \delta_2\mathcal{F} = 0$$

- Euler-Lagrange equation

$$\left[\left(\nabla^2 + 4H^2 - 2K\right)\frac{\partial}{\partial(2H)} + \left(\nabla\cdot\tilde{\nabla} + 2KH\right)\frac{\partial}{\partial K} - 2H\right]\mathcal{E} + \Delta p = 0.$$

$$\mathcal{E} = \frac{k_c}{2}(2H + c_0)^2 + \bar{k}K + \lambda$$

$$\Delta p - 2\lambda H + k_c\nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0 H - 2K) = 0$$

- Second order variation

if $\partial \mathcal{E}/\partial K = \bar{k} = const.$

$$\begin{aligned}
\delta^2 \mathcal{F} &= (\delta_1^2 + \delta_2^2 + \delta_3^2 + 2\delta_1\delta_2 + 2\delta_1\delta_3 + 2\delta_2\delta_3) \\
&= \int_M \Omega_3^2 \left[(4H^2 - 2K)^2 \frac{\partial^2 \mathcal{E}_H}{\partial (2H)^2} - 4HK \frac{\partial \mathcal{E}_H}{\partial (2H)} + 2K\mathcal{E}_H - 2Hp \right] dA \\
&\quad + \int_M \Omega_3 \nabla^2 \Omega_3 \left[4H \frac{\partial \mathcal{E}_H}{\partial (2H)} + 4(2H^2 - K) \frac{\partial^2 \mathcal{E}_H}{\partial (2H)^2} - \mathcal{E}_H \right] dA \\
&\quad - \int_M \frac{4\partial \mathcal{E}_H}{\partial (2H)} \Omega_3 \nabla \cdot \tilde{\nabla} \Omega_3 dA + \int_M \frac{\partial^2 \mathcal{E}_H}{\partial (2H)^2} (\nabla^2 \Omega_3)^2 dA \\
&\quad + \int_M \frac{\partial \mathcal{E}_H}{\partial (2H)} \left[\nabla (2H\Omega_3) \cdot \nabla \Omega_3 - 2\nabla \Omega_3 \cdot \tilde{\nabla} \Omega_3 \right] dA
\end{aligned}$$

$$\mathcal{E}_H = \mathcal{E} - \bar{k}K$$

Variational problem on open surface

- Functional

$$\mathcal{F} = \int_M \mathcal{E}(2H[\mathbf{r}], K[\mathbf{r}])dA + \int_C \Gamma(k_n, k_g)ds$$

- Euler-Lagrange equation

Equilibrium surface equation

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial(2H)} + (\nabla \cdot \tilde{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 2H\mathcal{E} = 0$$