shape equation and in-plane strain equations

- Shape equation

$$p - 2H(\lambda + k_d J) + k_c(2H + c_0)(2H^2 - c_0 H - 2K)$$
$$+ k_c \nabla^2 (2H) - \frac{k_d}{2}(a\varepsilon_{11} + 2b\varepsilon_{12} + c\varepsilon_{22}) = 0$$

- In-plane strain equations

$$k_d[-d(2J) \wedge \omega_2 - \frac{1}{2}(\varepsilon_{11} d\omega_2 - \varepsilon_{12} d\omega_1) + \frac{1}{2} d(\varepsilon_{12}\omega_1 + \varepsilon_{22}\omega_2)] = 0$$
$$k_d[d(2J) \wedge \omega_1 - \frac{1}{2}(\varepsilon_{12} d\omega_2 - \varepsilon_{22} d\omega_1) - \frac{1}{2} d(\varepsilon_{11}\omega_1 + \varepsilon_{12}\omega_2)] = 0$$

- Special example: spherical cell membrane with homogenous strains

$$\varepsilon_{12} = 0,\ \varepsilon_{11} = \varepsilon_{22} = \varepsilon = \text{constant};$$

$$pR^2 + (2\lambda + 3k_d\varepsilon)R + k_c c_0(c_0 R - 2) = 0$$

R is the radius of the spherical surface

Mechanical stability

- 2nd order variation of free energy (sphere)

$$\Omega_1\omega_1 + \Omega_2\omega_2 = d\Omega + *d\chi$$ (Hodge decomposed theorem)

$$\delta^2\mathcal{F} = G_1 + G_2$$

$$\begin{aligned} G_1 &= \int \Omega_3^2(3k_d/R^2 + 2k_c c_0/R^3 + p/R)dA \\ &+ \int \Omega_3\nabla^2\Omega_3(k_c c_0/R + 2k_c/R^2 + pR/2)dA + \int k_c(\nabla^2\Omega_3)^2 dA \\ &+ \frac{3k_d}{R}\int \Omega_3\nabla^2\Omega dA + k_d\int \left(\nabla^2\Omega\right)^2 dA + \frac{k_d}{2R^2}\int \Omega\nabla^2\Omega dA \\ G_2 &= \frac{k_d}{4}\int(\nabla^2\chi)^2 dA + \frac{k_d}{2R^2}\int \chi\nabla^2\chi dA \geq 0 \end{aligned}$$

- Expansion of spherical harmonic functions

$$\Omega_3 = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta, \phi),\ a_{lm}^* = (-1)^m a_{l,-m}$$

$$\Omega = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} b_{lm} Y_{lm}(\theta, \phi),\ b_{lm}^* = (-1)^m b_{l,-m}$$

$$\begin{aligned} G_1 \ = \ & \sum_{l=0}^{\infty} \sum_{m=0}^{l} 2|a_{lm}|^2 \{3k_d + [l(l+1) - 2][l(l+1)k_c/R^2 - k_c c_0/R - pR/2] \\ & - \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{3k_d}{R} l(l+1)(a_{lm}^* b_{lm} + a_{lm} b_{lm}^*) \\ & + \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{k_d}{R^2} \left[2l^2(l+1)^2 - l(l+1)\right] |b_{lm}|^2 \end{aligned}$$

Quadratic form

Critical pressure

If $p < p_l = \frac{3k_d}{[2l(l+1)-1]R} + \frac{2k_c[l(l+1)-c_0R]}{R^3} \quad (l = 2, 3, \cdots)$

then G_1 is positive definite.

$$p_c = \min\{p_l\} = \begin{cases} \frac{3k_d}{11R} + \frac{2k_c[6-c_0R]}{R^3} < \frac{k_c[23-2c_0R]}{R^3}, & (3k_dR^2 < 121k_c) \\ \frac{2\sqrt{3k_dk_c}}{R^2} + \frac{k_c}{R^3}(1-2c_0R), & (3k_dR^2 > 121k_c) \end{cases}$$

- k_d=0, $p_c = \frac{2k_c(6-c_0R)}{R^3}$, ~ spherical lipid vesicle [Ou-Yang & Helfrich 1987 *PRL* 59 2486]

- Classic shell: c_0=0, k_d~Yh, k_c~Yh^3, R>>h, p_c~Yh^2/R^2

- Membrane skeleton enhances the mechanical stability of cell membranes, at least for spherical shape

Taking typical data of cell membrane,
$k_c \sim 20k_BT$ [Duwe *et al.* 1990 *J. Phys. Fr.* **51** 945],
$k_d \sim 6\times10^{-4}k_BT/nm^2$ [Lenormand *et al.* 2001 *Biophys. J.* **81** 43],
$h \sim 4nm$, $R \sim 1\mu m$, $c_0R \sim 1$, we have $p_c \sim 2$ Pa.
If not considering k_d, we have $p_c \sim 0.2$ Pa.

Summary

- Several problems in the elasticity of biomembranes, smectic-A liquid crystal, and carbon related structures are discussed
- Variational problems on 2D surface are dealt with exterior differential forms
- Elasticity and stability of lipid bilayer and cell membrane are calculated and compared with each other.

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Thank you for your attention!