Geometric chaos and integrable vector fields in $\mathbb{R}^n$

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$x \in \mathbb{R}^n$, $X \in C^\infty(\mathbb{R}^n)$ (or C^ω)

$$\dot{x} = X(x)$$

Solution: $\phi(t; x_0)$ $(t \in \mathbb{R}) \Longrightarrow$ orbit on $\mathbb{R}^n$.

First integral: $f : \mathbb{R}^n \to \mathbb{R}$ such that $X(f) = X\nabla f = 0$.

Symmetry vector: S such that $[S, X] = \lambda X$ for some smooth function $\lambda : \mathbb{R}^n \to \mathbb{R}$.

Invariant set: $\Sigma \subset \mathbb{R}^n$ such that $\phi(t; \Sigma) \subset \Sigma$ for all $t \in \mathbb{R}$.

Some types of orbits: critical points, periodic, quasi-periodic and dense.

Topological boundary $\mathcal{F} = \{P \in \mathbb{R}^n$ such that for any neighborhood $N(P)$ of P there are points $Q, Q' \in N(P)$ for which the orbits of X through Q and Q' are bounded and unbounded respectively$\}$.

Geometric chaos: bounded and unbounded orbits are both dense on some open subset of $\mathbb{R}^n$, and hence $\mathcal{F}$ is an open set.

Other types of complex boundaries: fractal and Cantorian.

Classical example: Arnold's diffusion ($g \geq 3$) $\Longrightarrow$ Cantorian boundaries.

Geometric chaos is possible even if X has $1 \leq r \leq n-2$ first integrals.

Let X be a vector field with r (independent) first integrals $f = (f_1, \ldots, f_r)$ such that the level sets of f on some open set $U \subset \mathbb{R}^n$ are tori T^{n-r}. Assume $X \neq 0$ on U and that the orbits of X on these tori are periodic or quasi-periodic depending on the values of f.

Let L_r be a properly embedded r-dimensional half-plane diffeomorphic to $[0, \infty)^r$ and which transversely intersects each tori in U just once. There is a smooth diffeomorphism $\Phi : \mathbb{R}^n \backslash L_r \to \mathbb{R}^n$, and hence we can define the vector field $\tilde{X}$ transformed of $X|_{\mathbb{R}^n \backslash L_r}$ under Φ.

$\tilde{X}$ has geometric chaos.

Let X be a Hamiltonian vector field on $\mathbb{R}^{2g}$. Assume X is **Liouville-integrable**, i.e. it has g independent first integrals $\{f_1, \ldots, f_g\}$ in involution and the Hamiltonian vector fields X_{f_i} are all complete.

Theorem: Geometric chaos is not possible.

If X is **Liouville-separable** then it can be integrated by quadratures. In the analytic category the boundary is a semianalytic set. $\mathcal{F}$ is formed by "interior" and "exterior" components.

Examples: Henon-Heiles potential, Stark effect, straight-line wire.

X is **completely integrable** if it has $n-1$ independent first integrals $f = (f_1, \ldots, f_{n-1})$. $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ defines a submersion, and its level sets are properly embedded curves diffeomorphic to S^1 or $\mathbb{R}$. Examples: vector fields with many symmetries.

Any (smooth or analytic) link can be the zero set of f (Miyoshi). f has cyclic orbits if and only if the second homotopy group of the leaf space is not trivial (extension of Smith's exact sequence).

Theorem: $\mathcal{F}$ is and unbounded closed set in $\mathbb{R}^n$, foliated by open orbits of f, and of dimension $1 \leq dim\mathcal{F} \leq n-1$. In particular geometric chaos is not possible.

Old open problem: if f is analytic can the boundary $\mathcal{F}$ be **fractal or Cantorian**?. The answer is yes.

Let f be a submersion with S^1 orbits. Define the set Σ homeomorphic to $[0, \infty)^m$ and non-differentiable at any point (Weierstrass-type set). Assume that Σ intersects just once each closed orbit on certain open set U. The complement of Σ in $\mathbb{R}^n$ is analytically diffeomorphic to $\mathbb{R}^n$ (Morrey-Grauert theory). The same happens if Σ is homeomorphic to $[0, \infty)^m \times T_\infty$, where T_∞ is the Cantor set. Transforming $f|_{\mathbb{R}^n \setminus \Sigma}$ via the analytic diffeomorphic we get a new analytic submersion with fractal and Cantorian boundary.

This construction does not yield polynomial submersions, in fact if f is polynomial the boundary $\mathcal{F}$ is semialgebraic (Jelonek).

$\mathcal{F}$ being fractal or Cantorian is not an structurally stable property in general.

Open problems:

1. Examples of submersions with S^1 orbits.

2. Is geometric chaos structurally stable?.

3. Analytical criteria for ensuring that $\mathcal{F}$ is a "nice" set.

4. Physically relevant examples of integrable systems exhibiting geometric chaos or fractal/Cantorian boundary.

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