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QUANTUM COMPLEX MINKOWSKI SPACE

Creation of scalar massive particle - event in 8-dimensional phase space described by positions and momenta. We assume that both mass and energy is positive. We do not assume that mass is constant.

Complex Minkowski space has the interpretation of phase space of scalar conformal particles and anti-particles (as shown by A. Odzijewicz).

We have constructed and described the structure of the C^*-algebra which is the quatum phase space of scalar massive conformal particle (quantum complex Minkowski space).

$M_{1,3} \cong \mathbb{R}^{1,3}$– the Minkowski space

$\mathbf{G_{conf}}$ – the conformal group :

- Poincaré transformations

-Dilatation:

$$(Dx)^\mu = \alpha x^\mu \quad \text{where} \quad \alpha \in \mathbb{R}_+,$$

-4-**accelerations**:

$$(A_c x)^\mu := (I \circ T_c \circ I)^\mu = \frac{x^\mu + x^2 c^\mu}{1 + 2x \cdot c + x^2 c^2}, \qquad c \in \mathbb{R}^{1,3}$$

$\mathcal{T} = (\mathbb{C}^4, \eta)$-the twistor space;
$\overline{M}_{1,3} := G(2, \mathcal{T}_0)$-the conformally compactified Minkowski space;

Two models of the phase space of the scalar conformal particles.:

$$1)\ T^*\overline{M}_{1,3}$$

$$2)\ \mathbb{M} := G(2, \mathcal{T}) = G(2, \mathcal{T}_0)^{\mathbb{C}}$$

— the complex conformally compactified Minkowski space;

$$SU(2,2)/\mathbb{Z}_4 \cong G_{conf}$$

$$\sigma : SU(2,2)/\mathbb{Z}_4 \longrightarrow Bij\mathbb{M}$$

$$\mathbb{M} = \mathbb{M}^{00} \cup \mathbb{M}^{0-} \cup \mathbb{M}^{0+} \cup \mathbb{M}^{-+} \cup \mathbb{M}^{--} \cup \mathbb{M}^{++}$$

$$\mathbb{M}^{kl} := \{z \in \mathbb{M} : \text{sign } \eta|_z = (k,l)\}, k,l = +,-.0$$

$$\mathbb{M}^{00} = \overline{M}_{1,3}$$

$$\widetilde{\mathbb{M}} := \mathbb{M}^{++} \cup \mathbb{M}^{-+} \cup \mathbb{M}^{--}$$

$$\varphi_\infty : \widetilde{\mathbb{M}} \ni w = \left\{ \begin{pmatrix} W\zeta \\ \zeta \end{pmatrix} : \ \zeta \in \mathbb{C}^2 \right\} \longmapsto W \in Mat_{2\times 2}(\mathbb{C})$$

$$W = X + iY = w^\mu \sigma_\mu = (x^\mu + iy^\mu)\sigma_\mu;$$

The Kähler symplectic form

$$\omega_\lambda(W, W^\dagger) := -i\lambda\partial\bar{\partial}\left(\log\det(\frac{W^\dagger - W}{2i})\right),$$

where $\lambda \in \mathbb{R} \setminus \{0\}$

$$(\mathbb{M}^{++}, \omega_\lambda)$$

The momentum map

$$J_\lambda : \widetilde{\mathbb{M}} \to su(2,2)^* \cong su(2,2)$$

$$J_\lambda(w) := i\lambda(\Pi_w - \Pi_{w^\perp}) = i\lambda(2\Pi_w - \mathbb{I}),$$

where $\Pi_w : \mathcal{T} \to \mathcal{T}$ and $\Pi_{w^\perp} : \mathcal{T} \to \mathcal{T}$ are

$$\mathcal{T} := w \oplus w^\perp$$

$$J_\lambda \circ (\varphi_\infty)^{-1}(W) =$$

$$i\lambda \begin{pmatrix} (W + W^\dagger)(W - W^\dagger)^{-1} & -2W(W - W^\dagger)^{-1}W^\dagger \\ 2(W - W^\dagger)^{-1} & -\sigma_0 - 2(W - W^\dagger)^{-1}W^\dagger \end{pmatrix}$$

(i) **4-momentum**

$$p^{\nu} = \lambda \frac{y^{\nu}}{y^2},$$

(ii) **relativistic angular momentum**

$$m_{\mu\nu} = x_{\mu} p_{\nu} - p_{\mu} x_{\nu},$$

(iii) **dilatation**

$$d = x^{\mu} p_{\mu},$$

(iv) **4-acceleration**

$$a_{\mu} = -2(x^{\nu} p_{\nu}) x_{\mu} + x^2 p_{\mu} - \lambda^2 \frac{p_{\mu}}{p^2},$$

$$\omega_\lambda = dx^\nu \wedge dp_\nu$$

$\mathbb{M}^{++}$ it is the phase space of the scalar massive conformal particle since it contains states (x^μ, p_ν) such that $p^2 > 0$, $p_0 > 0$.

$\mathbb{M}^{--}$- is the phase space of the scalar massive conformal antiparticle,
$\mathbb{M}^{-+}$- is the phase space of the scalar tachyon.

$$\mathbb{T} := \varphi_\infty(\mathbb{M}^{++}) = \left\{ W \in Mat_{2\times 2}(\mathbb{C}) : \quad \frac{W - W^\dagger}{2i} > 0 \right\}$$

$$\mathbb{D} := \{ Z \in Mat_{2\times 2}(\mathbb{C}) : \quad \sigma_0 - Z^\dagger Z > 0 \}$$

Cayley transformation

$$\mathbb{T} \ni W \longmapsto \mathcal{C}(W) = (W + i\sigma_0)(W - i\sigma_0)^{-1} \in \mathbb{D}$$

$$\mathbb{D} \ni Z \longmapsto \mathcal{C}^{-1}(Z) = i(Z + \sigma_0)(Z - \sigma_0)^{-1} \in \mathbb{T}$$

THE COHERENT STATE MAP

We will require that the coherent state map $\mathcal{K}_\lambda : \mathbb{M}^{++} \to \mathbb{CP}(\mathcal{H})$ satisfies the following conditions:

i) is symplectic $\mathcal{K}_\lambda^* \omega_{FS} = \omega_\lambda$;

ii) there exists a unitary irreducible representation $U_\lambda : SU(2,2)/\mathbb{Z}_4 \to$ Aut($\mathcal{H}$) with respect to which a map $\mathcal{K}_\lambda$ is equivariant;

$$\begin{array}{ccc} \mathbb{M}^{++} & \xrightarrow{\mathcal{K}_\lambda} & \mathbb{CP}(\mathcal{H}) \\ \downarrow \sigma_g & & \downarrow [U_\lambda(g)] \\ \mathbb{M}^{++} & \xrightarrow{\mathcal{K}_\lambda} & \mathbb{CP}(\mathcal{H}) \end{array} \qquad \forall_{g \in SU(2,2)/\mathbb{Z}_4}$$

iii) $\mathcal{K}_\lambda$ is compatible with the choice of Kähler polarization $\left(\frac{\partial}{\partial \bar{w}^\mu}\right)_{\mu=0,1,2,3}$ on $\mathbb{M}^{++}$, i.e. it is holomorphic map.

Let $\mathcal{H}$ be the separable Hilbert space with the fixed

$$\left\{ \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle \right\}$$

orthonormal

$$\left\langle \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \middle| \begin{matrix} j' & m' \\ j_1' & j_2' \end{matrix} \right\rangle = \delta_{jj'}\delta_{mm'}\delta_{j_1 j_1'}\delta_{j_2 j_2'}$$

basis indexed by $m, 2j, 2j_1, 2j_2 \in \mathbb{N} \cup \{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$.

We will define now a map

$$K_\lambda : \mathbb{M}^{++} \cong \mathbb{D} \longrightarrow \mathcal{H} \setminus \{0\}$$

as follows

$$K_\lambda(Z) := |Z; \lambda\rangle := \sum_{j,m,j_1,j_2} \Delta^{jm}_{j_1 j_2}(Z) \left| \begin{matrix} j & m \\ j_1 & j_2 \end{matrix} \right\rangle ,$$

where coefficients $\Delta^{jm}_{j_1 j_2}(Z)$ are polynomials

$$\Delta^{jm}_{j_1 j_2}(Z) := (N^{\lambda}_{jm})^{-1} (\det Z)^m \sqrt{\frac{(j+j_1)!(j-j_1)!}{(j+j_2)!(j-j_2)!}} \times$$

$$\times \sum_{\substack{S \geqslant \max\{0, j_1+j_2\} \\ 0 \leqslant S \leqslant \min\{j+j_1, j+j_2\}}} \binom{j+j_2}{S} \binom{j-j_2}{S-j_1-j_2} z_{11}^{S} z_{12}^{j+j_1-S} z_{21}^{j+j_2-S} z_{22}^{S-j_1-j_2}$$

$$\mathcal{K}_\lambda := [K_\lambda] : \mathbb{M}^{++} \longrightarrow \mathbb{CP}(\mathcal{H}),$$

$$U_\lambda(g)\,|Z;\lambda\rangle := (\det(CZ+D))^{-\lambda}\,|\sigma_g(Z);\lambda\rangle\,,$$

where $\sigma_g(Z) = (AZ+B)(CZ+D)^{-1}$;
$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)$ and $3 < \lambda \in \mathbb{N}$.

Quantum polarization

$$f : \mathbb{D} \to \mathbb{C}$$

$$a(f)\,|Z\rangle = f(Z)\,|Z\rangle$$

$$a : \mathcal{O}^{++}(\mathbb{D}) \longrightarrow L^{\infty}(\mathcal{H})$$

Proposition .1. *The Banach algebra* $\mathcal{O}^{++}(\mathbb{D})$ *coincides with the Banach algebra* $H^{\infty}(\mathbb{D})$*.*

$$\mathcal{P}^{++} := a(H^{\infty}(\mathbb{D}))$$

- quantum holomorphic polarization

$$f_{kl}(Z) = z_{kl}; \quad k,l = 1,2$$

$$a_{kl} := a(f_{kl})$$

$$\begin{aligned} a_{11}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle &= \sqrt{\frac{(j-j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle \\ &+\sqrt{\frac{(j+j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle, \end{aligned} \tag{1}$$

$$\begin{aligned} a_{12}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle &= -\sqrt{\frac{(j-j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle \\ &+\sqrt{\frac{(j+j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle, \end{aligned} \tag{2}$$

$$a_{21}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = -\sqrt{\frac{(j+j_1+1)(j-j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle$$
$$+\sqrt{\frac{(j-j_1)(j+j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\rangle, \qquad (3)$$

$$a_{22}\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle = \sqrt{\frac{(j+j_1+1)(j+j_2+1)m}{(2j+1)(2j+2)(m+\lambda-2)}}\left|\begin{matrix} j+\frac{1}{2} & m-1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle$$
$$+\sqrt{\frac{(j-j_1)(j-j_2)(m+2j+1)}{(m+2j+\lambda-1)2j(2j+1)}}\left|\begin{matrix} j-\frac{1}{2} & m \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\rangle. \qquad (4)$$

In formulae above one should put $\left|\begin{matrix} j & m \\ j_1 & j_2 \end{matrix}\right\rangle := 0$ if indices do not satisfy the conditions $m, 2j \in \mathbb{N} \cup \{0\}$ and $-j \leqslant j_1, j_2 \leqslant j$.

$$\mathcal{P}_{pol}^{++} := \overline{\langle I, a_{11}, a_{12}, a_{21}, a_{22} \rangle}$$

— quantum algebric polarization

$$\overline{\mathcal{P}_{pol}^{++}} := \overline{\langle I, a_{11}^*, a_{12}^*, a_{21}^*, a_{22}^* \rangle}$$

Proposition .2.

(i) $\overline{\mathcal{P}_{pol}^{++}}$ is isometrically isomorphic with Banach algebra $\overline{Pol(\bar{\mathbb{D}})}$ obtained by closing the algebra $Pol(\bar{\mathbb{D}})$ in norm $\|\cdot\|_{\mathrm{sup}}$ in the space $C(\bar{\mathbb{D}})$, i.e. $a(f) \in \mathcal{P}_{pol}^{++}$ if and only if f is holomorphic on $\mathbb{D}$ and is continuous on closure $\bar{\mathbb{D}}$.

(ii) The space of maximal ideals (spectrum) of the algebra $\mathcal{P}_{pol}^{++}$ is homeomorphic with $\bar{\mathbb{D}}$.

(iii) $\mathcal{P}_{pol}^{++}$ is semisimple Banach algebra, i.e. its radical is trivial.

(iv) $\mathcal{P}_{pol}^{++}$ is proper subalgebra of Banach algebra $\mathcal{P}^{++}$, i.e. one has $\mathcal{P}_{pol}^{++} \subsetneq \mathcal{P}^{++}$.

(v) Vacuum state $|0\rangle$ is a cyclic vector for Banach algebra $\overline{\mathcal{P}_{pol}^{++}}$.

Notation

$$\mathbb{A} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{P}_{pol}^{++} \otimes Mat_{2\times 2}(\mathbb{C}).$$

$$\mathbb{U}_\lambda(g^{-1})\mathbb{A}\mathbb{U}_\lambda(g) := \begin{pmatrix} U_\lambda(g^{-1})a_{11}U_\lambda(g) & U_\lambda(g^{-1})a_{12}U_\lambda(g) \\ U_\lambda(g^{-1})a_{21}U_\lambda(g) & U_\lambda(g^{-1})a_{22}U_\lambda(g) \end{pmatrix}.$$

Action of conformal group on Banach algebra $\mathcal{P}_{pol}^{++}$.

Proposition .3. *For any* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)_{\mathbb{D}}$ *we have:*

(i)

$$\sigma_g(\mathbb{A}) := (A\mathbb{A} + B)(C\mathbb{A} + D)^{-1} \in \mathcal{P}_{pol}^{++} \otimes Mat_{2\times 2}(\mathbb{C})$$

(ii)

$$\mathbb{U}_\lambda(g^{-1})\mathbb{A}\mathbb{U}_\lambda(g) = \sigma_g(\mathbb{A})$$

Quantum Complex Minkowski Space

Definition .4. C^***-algebra with polarization** is a pair $(\mathcal{A}, \mathcal{P})$ consisting of unital C^*-algebra $\mathcal{A}$ and its commutative Banach subalgebra $\mathcal{P}$ such that:

(i) $\mathcal{P}$ generates $\mathcal{A}$, i.e. $\mathcal{A}$ is the smallest C^*-algebra containing $\mathcal{P}$;

(ii) $\mathcal{P} \cap \overline{\mathcal{P}} = \mathbb{C}\mathbb{I}$.

Subalgebra $\mathcal{P}$ will be called a quantum polarization of algebra $\mathcal{A}$.

$$(\mathcal{M}^{++}, \mathcal{P}^{++}), \ (\mathcal{M}^{++}_{pol}, \mathcal{P}^{++}_{pol}); \ \mathcal{M}^{++} := C^*(\mathcal{P}^{++}), \quad \mathcal{M}^{++}_{pol} := C^*(\mathcal{P}^{++}_{pol})$$

Proposition .5.

(i) *Autorepresentation of the* C^**-algebra* $\mathcal{M}_{pol}^{++}$ *in* $L^\infty(\mathcal{H})$ *is irreducible in* $\mathcal{H}$ *and* $\mathcal{P}_{pol}^{++} \cap \overline{\mathcal{P}}_{pol}^{++} = \mathbb{C}I$*.*

(ii) $\mathcal{M}_{pol}^{++}$ *is weakly (strongly) dense in* $L^\infty(\mathcal{H})$*.*

(iii) $\mathcal{M}_{pol}^{++}$ *contains the ideal* $L^0(\mathcal{H})$ *of compact operators. Thus all ideals of* C^**-algebra* $\mathcal{M}_{pol}^{++}$ *with irreducible autorepresentation in* $\mathcal{H}$ *also contains* $L^0(\mathcal{H})$*.*

(iv) $\mathcal{M}_{pol}^{++}$ *is conformally invariant, i.e.*

$$\forall\, g \in SU(2,2) \quad U_\lambda^*(g)\mathcal{M}_{pol}^{++}U_\lambda(g) \subset \mathcal{M}_{pol}^{++}.$$

(v) $\mathcal{P}_{pol}^{++} \cap L^0(\mathcal{H}) = \{0\}$.

(vi) Commutator ideal $\mathcal{C}omm\mathcal{M}_{pol}^{++}$ *of algebra* $\mathcal{M}_{pol}^{++}$ *contains* $L^0(\mathcal{H})$ *but is not equal* $L^0(\mathcal{H}) \subsetneq \mathcal{C}omm\mathcal{M}_{pol}^{++}$.

(vii) Points (i), (ii), (iii), (v), (vi) are true also for $\mathcal{M}^{++}$ *and* $\mathcal{P}^{++}$.

Theorem .6. *There is the following exact sequence of homomorphisms of* C^**-algebras*

$$0 \longrightarrow \mathcal{C}omm\mathcal{M}_{pol}^{++} \xrightarrow{\lambda} \mathcal{M}_{pol}^{++} \xrightarrow{p_\lambda} C(\mathbb{M}^{00}) \longrightarrow 0,$$

where $C(\mathbb{M}^{00})$ *is the* C^**-algebra of continuous functions on the conformally compactified Minkowski space* $\mathbb{M}^{00}$.

$$\mathcal{M}_{pol}^{++}/\mathcal{C}omm\mathcal{M}_{pol}^{++} \cong C(\mathbb{M}^{00})$$

$$\mathrm{spec}(C(\mathbb{M}^{00})) \cong \mathbb{M}^{00}$$

Definition .7. The symbol $\mathcal{A}^{++}$ will denote the vector space of the linear operators on $\mathcal{H}$ with domain containing $\mathcal{L}(K_\lambda(\mathbb{M}^{++}))$ and such that their Berezin symbols are real analytical functions on $\mathbb{D}$.

We will equip $\mathcal{A}^{++}$ with a natural topology.

Definition .8. We will say that a sequence of operators $\{A_n\} \subset \mathcal{A}^{++}$ converges in the weak-coherent topology to an operator $A \in \mathcal{A}^{++}$ if

$$\lim_{n \to \infty} \langle Z|A_n|V\rangle = \langle Z|A|V\rangle$$

for all coherent states $|Z\rangle$, $|V\rangle$.

$$f(Z^\dagger, Z) = \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}} \bar{z}_{11}^{i_{11}} \bar{z}_{12}^{i_{12}} \bar{z}_{21}^{i_{21}} \bar{z}_{22}^{i_{22}} z_{11}^{j_{11}} z_{12}^{j_{12}} z_{21}^{j_{21}} z_{22}^{j_{22}}.$$

The infinite sum

$$\begin{aligned} Q_\lambda(f) := & \sum f_{i_{11},i_{12},i_{21},i_{22},j_{11},j_{12},j_{21},j_{22}} {a_{11}^\dagger}^{i_{11}} {a_{12}^\dagger}^{i_{12}} {a_{21}^\dagger}^{i_{21}} {a_{22}^\dagger}^{i_{22}} a_{11}^{i_{11}} a_{12}^{i_{12}} a_{21}^{i_{21}} a_{22}^{i_{2} \\ = & : f(\mathbb{A}^\dagger, \mathbb{A}) :, \end{aligned} \tag{5}$$

considered in weak-coherent topology, defines a linear operator $Q_\lambda(f) \in \mathcal{A}^{++}$ with Berezin symbol f belonging to $\mathcal{A}(\mathbb{D})$.

$*$-product

$$(f *_\lambda g)(Z^\dagger, Z) := \frac{\langle Z| : f(\mathbb{A}^\dagger, \mathbb{A}) : : g(\mathbb{A}^\dagger, \mathbb{A}) : |Z\rangle}{\langle Z|Z\rangle},$$

where $f, g \in \mathcal{A}(\mathbb{D})$.

Properties:

$$Q_\lambda(f *_\lambda g) = Q_\lambda(f) Q_\lambda(g)$$

$$Q_\lambda(\bar{f}) = Q_\lambda(f)^*$$

$$\langle Q_\lambda(f)\rangle = f,$$

where $\langle F\rangle (Z^\dagger, Z) := \frac{\langle Z|FZ\rangle}{\langle Z|Z\rangle}$

$$\langle U_\lambda(g)\rangle(Z^\dagger, Z) = (\mathrm{det}(CZ + D))^{-\lambda} \left(\frac{\mathrm{det}(E - Z^\dagger \sigma_g(Z))}{\mathrm{det}(E - Z^\dagger Z)}\right)^{-\lambda}$$

$$U_\lambda(g) = Q_\lambda(\langle U_\lambda(g)\rangle) = :\!\left(\frac{\mathrm{det}(E - \mathbb{A}^\dagger \sigma_g(\mathbb{A}))}{\mathrm{det}(E - \mathbb{A}^\dagger \mathbb{A})}\right)^{-\lambda}\!:\; (\mathrm{det}(C\mathbb{A}+D))^{-\lambda},$$

(i) 4-momentum

$$Q_\lambda(p_\mu) = i\lambda : (\mathrm{det}(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\,\mathrm{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) :, \tag{6}$$

(ii) relativistic angular momentum

$$Q_\lambda(m_{\mu\nu}) = i\lambda\left(\frac{1}{2}\,\mathrm{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\mathrm{det}(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\,\mathrm{Tr}(\sigma_\nu(\mathbb{W} - \mathbb{W}^\dagger)) : - \right.$$
$$\left. -\frac{1}{2}\,\mathrm{Tr}(\sigma_\nu\mathbb{W}^\dagger) : (\mathrm{det}(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\,\mathrm{Tr}(\sigma_\mu(\mathbb{W} - \mathbb{W}^\dagger)) :\right), \tag{7}$$

(iii) dilatation

$$Q_\lambda(d) = i\lambda\,\mathrm{Tr}(\sigma_\mu\mathbb{W}^\dagger) : (\mathrm{det}(\mathbb{W} - \mathbb{W}^\dagger))^{-1}\,\mathrm{Tr}(\sigma^\mu(\mathbb{W} - \mathbb{W}^\dagger)) : -2i\lambda\mathbb{I}, \tag{8}$$

(iv) 4-acceleration

$$
\begin{aligned}
Q_\lambda(a_\nu) = i\lambda\,\mathsf{det}(\mathbb{W}^\dagger) : (\mathsf{det}(\mathbb{W}-\mathbb{W}^\dagger))^{-1}\,\mathsf{Tr}(\sigma_\nu(\mathbb{W}-\mathbb{W}^\dagger)) : - \\
- i\lambda\frac{1}{2}\,\mathsf{Tr}(\sigma_\nu\mathbb{W}^\dagger)\,\mathsf{Tr}(\sigma^\beta\mathbb{W}^\dagger) : (\mathsf{det}(\mathbb{W}-\mathbb{W}^\dagger))^{-1}\,\mathsf{Tr}(\sigma_\beta(\mathbb{W}-\mathbb{W}^\dagger \\
+ i\lambda\,\mathsf{Tr}(\sigma_\nu\mathbb{W}^\dagger),
\end{aligned} \tag{9}
$$

where $(\mathbb{W}^\dagger, \mathbb{W})$ are matrix quantum coordinates obtained from $(\mathbb{A}^\dagger, \mathbb{A})$ by **Cayley transform**

$$\mathbb{W} = i(\mathbb{A} + E)(\mathbb{A} - E)^{-1},$$

which is well defined in the weak-coherent topology.

Proposition .9. *The quantization of the canonical coordinates* (x^ν, p_μ) *gives the operators* $(Q_\lambda(x^\mu), Q_\lambda(p_\nu))$ *satisfying the following commutation relations*

$$[Q_\lambda(x^\mu), Q_\lambda(x^\nu)] = 0,$$

$$[Q_\lambda(p_\mu), Q_\lambda(p_\nu)] = 0,$$

$$[Q_\lambda(x^\mu), Q_\lambda(p_\nu)] = -i\delta^\mu_\nu 1.$$

The representation the quantum observables in $L^2\mathcal{O}(\mathbb{T}, \tilde{d\mu_\lambda})$:

$$\widehat{p_\mu} = -i\frac{\partial}{\partial w^\mu},$$

$$\widehat{m_{\mu\nu}} = -i(w_\mu\frac{\partial}{\partial w^\nu} - w_\nu\frac{\partial}{\partial w^\mu}),$$

$$\widehat{d} = -2iw^\mu\frac{\partial}{\partial w^\mu} - 2i\lambda,$$

$$\widehat{a}_\nu = -iw^2(\delta_\nu^\beta - 2w_\nu w^\beta)\frac{\partial}{\partial w^\beta} + 2i\lambda w_\nu,$$