Construction of Elliptic Solutions to the Quintic Complex One-dimensional Ginzburg-Landau Equation

S. Yu. Vernov
Skobeltsyn Institute of Nuclear Physics, Moscow State University,
Moscow, Russia

Varna, 12.06.2006

S.Yu. Vernov, nlin.PS/0602060

In 2003 R. Conte and M. Musette have proposed way to search for elliptic and degenerate elliptic solutions to a polynomial autonomous differential equation.

Let us reformulate this method for a system of such equations:

$$F_i(\tilde{\mathbf{y}}_{;t}^{(n)}, \tilde{\mathbf{y}}_{;t}^{(n-1)}, \dots, \tilde{\mathbf{y}}_{;t}, \tilde{\mathbf{y}}) = 0, \qquad i = 1, \dots, N, \qquad (1)$$

where $\tilde{\mathbf{y}} = \{y_1(t), y_2(t), \dots, y_L(t)\}$ and $y_{j;t}^{(k)} = \dfrac{\mathrm{d}^k y_j}{\mathrm{d}t^k}$.

F_i is a polynomial.

Any elliptic function (including any degenerate one) is a solution of some first order polynomial autonomous differential equation.

The classical results of P. Painlevé, L. von Fuchs, C.A.A. Briot and J.-C. Bouquet allow one to construct the suitable form of an equation, whose general solution is a meromorphic function with poles of order p:

$$\sum_{k=0}^{m} \sum_{j=0}^{(p+1)(m-k)/p} h_{j,k} y^j {y_t}^k = 0, \qquad h_{0,m} = 1, \tag{2}$$

in which m is a positive integer number and $h_{j,k}$ are constants to be determined.

The general solution of (2) is either an elliptic function, or a rational function of $e^{\gamma x}$, or a rational function of x.

The Conte–Musette algorithm is the following:

1. Choose a positive integer number m.
2. Construct solutions of system (1) in the form of Laurent series. One should compute more coefficients of the Laurent series than the number of numerical parameters in the Laurent series plus the number of $h_{j,k}$.
3. Choose a Laurent series expansion for some function y_k and substitute the obtained Laurent series coefficients into Eq. (2). This substitution transforms (2) into a linear and overdetermined system in $h_{j,k}$ with coefficients depending on numerical parameters.
4. Eliminate coefficients $h_{j,k}$ and get a system in parameters.
5. Solve the obtained nonlinear system.

1 Properties of the elliptic functions

Let us recall some definitions and theorems.

The function $\varrho(z)$ of the complex variable z is a doubly-periodic function if there exist two numbers ω_1 and ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$, such that for all $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z + \omega_1) = \varrho(z + \omega_2). \tag{3}$$

By definition a double-periodic meromorphic function is called an elliptic function. These periods define the period parallelograms with vertices z_0, $z_0+N_1\omega_1$, $z_0+N_2\omega_2$ and $z_0+N_1\omega_1+N_2\omega_2$, where N_1 and N_2 are arbitrary natural numbers and z_0 is an arbitrary complex number. The fundamental parallelogram of periods is called a parallelogram of period, which does not include other parallelogram of periods, that corresponds to $N_1 = N_2 = 1$.

The classical theorems for elliptic functions prove that

- If an elliptic function has no poles then it is a constant.
- The number of elliptic function poles within any finite period parallelogram is finite.
- The sum of residues within any finite period parallelogram is equal to zero (**the residue theorem**).
- If $\varrho(z)$ is an elliptic function then any rational function of $\varrho(z)$ and its derivatives is an elliptic function as well.
- For each elliptic function $\varrho(z)$ there exist such m $(m \geqslant 2)$ and such coefficients $h_{i,j}$ that $\varrho(z)$ is a solution of Eq. (2).

Lemma 1 *An elliptic function can not have two poles with the same Laurent series expansions in its fundamental parallelogram of periods.*

Proof.

Let some elliptic function $\varrho(\xi)$ has two poles in points ξ_0 and ξ_1, which belong to the fundamental parallelogram of periods. The corresponding Laurent series are the same and have the convergence radius R. Then the function $\upsilon(\xi) = \varrho(\xi - \xi_0) - \varrho(\xi - \xi_1)$ is an elliptic function as a difference between two elliptic functions with the same periods. At the same time for all ξ such that $|\xi| < R$ $\upsilon(\xi) = 0$, therefore, $\upsilon(\xi) \equiv 0$ and $\varrho(\xi - \xi_0) \equiv \varrho(\xi - \xi_1)$ and $\xi_1 - \xi_0$ is a period of $\varrho(\xi)$. It contradicts to our assumption that both points ξ_0 and ξ_1 belong to the fundamental parallelogram of periods.

2 Construction of elliptic solutions

2.1 The quintic complex Ginzburg–Landau equation

The one-dimensional quintic complex Ginzburg–Landau equation (CGLE5) is as follows

$$\mathrm{i}\mathcal{A}_t + p\mathcal{A}_{xx} + q|\mathcal{A}|^2\mathcal{A} + r|\mathcal{A}|^4\mathcal{A} - \mathrm{i}\gamma\mathcal{A} = 0, \tag{4}$$

where $\mathcal{A}_t \equiv \dfrac{\partial \mathcal{A}}{\partial t}$, $\mathcal{A}_{xx} \equiv \dfrac{\partial^2 \mathcal{A}}{\partial x^2}$, $p, q, r \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

One of the most important directions in the study of the CGLE5 is the consideration of its travelling wave reduction:

$$\mathcal{A}(x,t) = \sqrt{M(\xi)}e^{\mathrm{i}(\varphi(\xi) - \omega t)}, \quad \xi = x - ct, \quad c, \omega \in \mathbb{R}. \tag{5}$$

Substituting (5) in (4) we obtain

$$2pM''M - pM'^2 + 4\mathrm{i}p\psi MM' + 2\Big(2\omega - \mathrm{i}c - 2\mathrm{i}\gamma + \\ + 2c\psi - 2p\psi^2 + 2\mathrm{i}p\psi'\Big)M^2 + 4qM^3 + 4rM^4 = 0, \tag{6}$$

where $\psi \equiv \varphi' \equiv \frac{d\varphi}{d\xi}$, $M' \equiv \frac{dM}{d\xi}$. Equation (6) is a system of two equations: both real and imaginary parts of its left-hand side have to be equal to zero:

$$\begin{cases} 2MM'' - M'^2 - 4M^2\tilde{\psi}^2 - 2\tilde{c}MM' + 4g_iM^2 + 4d_rM^3 + 4u_rM^4 = 0, \\ M\tilde{\psi}' + \tilde{\psi}\left(M' - \tilde{c}M\right) - g_rM + d_iM^2 + u_iM^3 = 0, \end{cases} \tag{7}$$

Note that to obtain (7) from (6) we assume that the functions

$M(\xi)$ and $\psi(\xi)$ are real. New real variables are as follows

$$u_r + \mathrm{i}u_i = \frac{r}{p}, \qquad d_r + \mathrm{i}d_i = \frac{q}{p}, \qquad s_r - \mathrm{i}s_i = \frac{1}{p}, \tag{8}$$

$$g_r + \mathrm{i}g_i = (\gamma + \mathrm{i}\omega)(s_r - \mathrm{i}s_i) + \frac{1}{2}c^2 s_i s_r + \frac{\mathrm{i}}{4}c^2 s_r^2, \tag{9}$$

and

$$\tilde{\psi} \equiv \psi - \frac{cs_r}{2}, \qquad \tilde{c} \equiv cs_i. \tag{10}$$

System (7) includes seven numerical parameters: g_r, g_i, d_r, d_i, u_r, u_i and $\tilde{c}$.

The standard way to construct exact solutions for system (7) is to transform it into the equivalent third order differential equation for M. We rewrite the first equation of system (7) as

$$\tilde{\psi}^2 = \frac{G}{M^2}, \tag{11}$$

where

$$G \equiv \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{\tilde{c}}{2}MM' + g_i M^2 + d_r M^3 + u_r M^4. \quad (12)$$

We can express $\tilde{\psi}$ in terms of M and its derivatives:

$$\tilde{\psi} = \frac{G' - 2\tilde{c}G}{2M^2\left(g_r - d_i M - u_i M^2\right)}, \quad (13)$$

and obtain the third order equation for M:

$$(G' - 2\tilde{c}G)^2 + 4GM^2(g_r - d_i M - u_i M^2)^2 = 0. \quad (14)$$

2.2 The Laurent series solutions

Below we consider the case

$$\frac{p}{r} \notin \mathbb{R}, \quad (15)$$

which corresponds to the condition $u_i \neq 0$. In this case Eq. (14) is not integrable and its general solution (which should depend on three arbitrary integration constants) is not known. Using the Painlevé analysis it has been shown that single-valued solutions of (7) can depend on only one arbitrary parameter. System (7) is autonomous, so this parameter is ξ_0: if $M = f(\xi)$ is a solution, then $M = f(\xi - \xi_0)$, where $\xi_0 \in \mathbb{C}$ has to be a solution.

All known exact solutions of (7) are elementary (rational or hyperbolic) functions.

The purpose of this section is to find an elliptic solution of (7).

System (7) is invariant under the transformation:

$$\tilde{\psi} \to -\tilde{\psi}, \quad g_r \to -g_r, \quad d_i \to -d_i, \quad u_i \to -u_i, \qquad (16)$$

therefore we can assume that $u_i > 0$ without loss of generality.

Moreover, using scale transformations:

$$M \rightarrow \lambda M, \quad d_r \rightarrow \frac{d_r}{\lambda}, \quad d_i \rightarrow \frac{d_i}{\lambda}, \quad u_r \rightarrow \frac{u_r}{\lambda^2}, \quad u_i \rightarrow \frac{u_i}{\lambda^2}, \tag{17}$$

we can always put $u_i = 1$.

Let us construct the Laurent series solutions to system (7). We assume that in a sufficiently small neighborhood of the singularity point ξ_0:

$$\tilde{\psi} = A(\xi - \xi_0)^{\alpha} \qquad \text{and} \qquad M = B(\xi - \xi_0)^{\beta}. \tag{18}$$

Substituting (18) into (7) we obtain that two or more terms in the equations of system (7) balance if and only if $\alpha = -1$ and $\beta = -1$. In other words in this case these terms have equal powers and the other terms can be ignored as $t \longrightarrow t_0$. We obtain values of A and B from the following algebraic system:

$$\begin{cases} B^2\left(3 - 4A^2 + 4u_r B^2\right) = 0, \\ 2A - B^2 = 0. \end{cases} \tag{19}$$

System (19) has four nonzero solutions:

$$A_1 = u_r + \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_1 = \sqrt{2u_r + \sqrt{4u_r^2 + 3}}, \tag{20}$$

$$A_2 = u_r + \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_2 = -\sqrt{2u_r + \sqrt{4u_r^2 + 3}}, \tag{21}$$

$$A_3 = u_r - \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_3 = \sqrt{2u_r - \sqrt{4u_r^2 + 3}} \tag{22}$$

and

$$A_4 = u_r - \frac{1}{2}\sqrt{4u_r^2 + 3}, \qquad B_4 = -\sqrt{2u_r - \sqrt{4u_r^2 + 3}}. \tag{23}$$

Therefore, system (7) has four types of the Laurent series solutions. Denote them as follows:

$$\tilde{\psi}_k = \frac{A_k}{\xi} + a_{k,0} + a_{k,1}\xi + \ldots, \qquad M_k = \frac{B_k}{\xi} + b_{k,0} + b_{k,1}\xi + \ldots, \tag{24}$$

where $k = 1..4$.

Let $M(\xi)$ is a nontrivial elliptic function. $\mapsto$

$\tilde{\psi}$ is a constant or a nontrivial elliptic function.

$\tilde{\psi}$ is a constant $\mapsto$ M is not a nontrivial elliptic function. $\mapsto$

$\tilde{\psi}$ is a nontrivial elliptic function and has poles.

Let us define a number of poles of $M(\xi)$ in its fundamental parallelogram of periods.

Let M has a pole of type M_1, hence, according to the residue theorem, it should has a pole of type M_2. So $\tilde{\psi}$ has poles with the Laurent series $\tilde{\psi}_1$ and $\tilde{\psi}_2$. As an elliptic function it should have a pole of type $\tilde{\psi}_3$ or $\tilde{\psi}_4$ as well. It means that the function $M(\xi)$ should have a pole of type M_3 and, hence, a pole of type M_4. So $M(\xi)$ should have at least four different poles in its the fundamental parallelogram of periods. Using **Lemma 1**, we obtain that the function $M(\xi)$ can not have the same poles in the fundamental parallelogram of periods. Therefore, $M(\xi)$ has exactly four poles in its fundamental parallelogram of periods.

By means of the residue theorem for $\tilde{\psi}$ we obtain

$$u_r = 0. \tag{25}$$

We obtain that the CGLE5 with $u_r \neq 0$ has no elliptic solution in the wave form. In the case $u_r = 0$ possible elliptic solutions should have four simple poles in the fundamental parallelogram of periods, and, therefore, has the following form:

$$M(\xi - \xi_0) = C + \sum_{k=1}^{4} B_k \zeta(\xi - \xi_k), \tag{26}$$

where the function $\zeta(\xi)$ is an integral of the Weierstrass elliptic function multiplied by -1: $\zeta'(\xi) = -\wp(\xi)$, C and ξ_k are constants to be defined.

To obtain restrictions on other parameters, we use the Hone method (2005) and apply the residue theorem to the functions $\tilde{\psi}^2$, $\tilde{\psi}^3$, and so on. The residue theorem for the function $\tilde{\psi}^2$ gives the equation:

$$\sum_{k=1}^{4} A_k a_{k,0} = 0. \tag{27}$$

The values of $a_{k,0}$ are as follows ($u_r = 0$):

$$a_{1,0} = \frac{\sqrt{3}}{48}\left(6\tilde{c} - \sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r\right), \tag{28}$$

$$a_{2,0} = \frac{\sqrt{3}}{48}\left(6\tilde{c} + \sqrt[4]{27}d_i + 15\sqrt[4]{3}d_r\right), \tag{29}$$

$$a_{3,0} = -\frac{\sqrt{3}}{48}\left(6\tilde{c} + \mathrm{i}\left(\sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r\right)\right), \tag{30}$$

$$a_{4,0} = -\frac{\sqrt{3}}{48}\left(6\tilde{c} - \mathrm{i}\left(\sqrt[4]{27}d_i - 15\sqrt[4]{3}d_r\right)\right). \tag{31}$$

Substituting A_k and $a_{k,0}$ in (27), we obtain

$$\sum_{k=1}^{4} A_k a_{k,0} = \frac{3}{4}\tilde{c} = 0, \tag{32}$$

therefore $\tilde{c} = 0$.

For the function $\tilde{\psi}^3$ the residue theorem gives

$$d_i{}^2 + 27 d_r{}^2 = 0 \quad \rightarrow \quad d_i = \pm \mathrm{i}\sqrt{27} d_r. \tag{33}$$

The parameters d_r and d_i should be real, therefore, $d_r = 0$ and $d_i = 0$. So, consideration of $\tilde{\psi}^2$ and $\tilde{\psi}^3$ gives three restrictions:

$$\tilde{c} = 0, \qquad d_r = 0 \qquad \text{and} \qquad d_i = 0. \tag{34}$$

The residue theorem for $\tilde{\psi}^4$ gives the restriction

$$g_i g_r = 0. \tag{35}$$

Taking into account (25) and (34) we obtain system (7) in the following form:

$$\begin{cases} 2MM'' - M'^2 - 4M^2\tilde{\psi}^2 + 4g_i M^2 = 0, \\ \tilde{\psi}' M + \tilde{\psi} M' - g_r M + M^3 = 0. \end{cases} \tag{36}$$

To find elliptic solutions to system (36) we use the Conte–Musette method. Equation (2) with $m = 1$ has no elliptic solution. Let $\tilde{\psi}(\xi)$ satisfies Eq. (2) with $m = 2$:

$$\begin{aligned} &\tilde{\psi}'^2 + \left(\tilde{h}_{2,1}\tilde{\psi}^2 + \tilde{h}_{1,1}\tilde{\psi} + \tilde{h}_{0,1}\right)\tilde{\psi}' + \\ &+ \tilde{h}_{4,0}\tilde{\psi}^4 + \tilde{h}_{3,0}\tilde{\psi}^3 + \tilde{h}_{2,0}\tilde{\psi}^2 + \tilde{h}_{1,0}\tilde{\psi} + \tilde{h}_{0,0} = 0. \end{aligned} \tag{37}$$

Substituting in (37) the Laurent series of $\tilde{\psi}$, which begins from A_1 (more exactly we use the first ten coefficients), we obtain the following solution $\tilde{h}_{k,j}$ for an arbitrary value of the parameter $g_r \neq 0$ and $g_i = 0$:

$$\tilde{h}_{4,0} = -\frac{4}{3},\ \tilde{h}_{0,0} = -\frac{g_r^2}{9},\ \tilde{h}_{3,0} = \tilde{h}_{2,0} = \tilde{h}_{1,0} = \tilde{h}_{0,1} = \tilde{h}_{1,1} = \tilde{h}_{2,1} = 0, \tag{38}$$

a few solutions with $g_i = 0$ and $g_r = 0$ and no solution for $g_i \neq 0$.

In the case of solutions (38) the function $\tilde{\psi}(\xi)$ satisfies the equation

$$\tilde{\psi}'^2 = \frac{4}{3}\tilde{\psi}^4 + \frac{g_r^2}{9}. \qquad (39)$$

The polynomial in the right hand side of (39) has four different roots, therefore $\tilde{\psi}$ is a non-degenerate elliptic function.

Surely we do not rigorously prove the existence of elliptic solutions to the CGLE5. For rigorous proof we should find the function $M(\xi)$ and check that this function is a solution of (14).

The function $M(\xi)$ in a parallelogram of periods has four different Laurent series expansions, so we choose $m = 4$. The general form of (2) for $m = 4$ and $p = 1$ is the following:

$$M'^4 + \left(h_{2,3}M^2 + h_{1,3}M + h_{0,3}\right) M'^3 +
+ \left(h_{4,2}M^4 + h_{3,2}M^3 + h_{2,2}M^2 + h_{1,2}M + h_{0,2}\right) M'^2 +
+ \left(h_{6,1}M^6 + h_{5,1}M^5 + h_{4,1}M^4 + h_{3,1}M^3 + h_{2,1}M^2 + h_{1,1}M +
+ h_{0,1}\right) M' + h_{8,0}M^8 + h_{7,0}M^7 + h_{6,0w}M^6 + h_{5,0}M^5 +
+ h_{4,0}M^4 + h_{3,0}M^3 + h_{2,0}M^2 + h_{1,0}M + h_{0,0} = 0. \tag{40}$$

Substituting the Laurent series M_k from (24), we transform the left hand side of (40) into the Laurent series, which has to be equal to zero. Therefore, we obtain the algebraic system in $h_{i,j}$ and g_r. The first algebraic equation, which corresponds to $1/\xi^8$ is

$$B_k^4 \left(h_{8,0}B_k^4 - h_{6,1}B_k^3 + h_{4,2}B_k^2 - h_{2,3}B_k + 1\right) = 0, \tag{41}$$

where B_k is defined by (20)–(23).

We seek only elliptic solutions, so we know that all B_k have to satisfy (41) and can consider Eq. (41) as the following system:

$$\begin{cases} h_{8,0}B_1^4 - h_{6,1}B_1^3 + h_{4,2}B_1^2 - h_{2,3}B_1 + 1 = 0, \\ h_{8,0}B_2^4 - h_{6,1}B_2^3 + h_{4,2}B_2^2 - h_{2,3}B_2 + 1 = 0, \\ h_{8,0}B_3^4 - h_{6,1}B_3^3 + h_{4,2}B_3^2 - h_{2,3}B_3 + 1 = 0, \\ h_{8,0}B_4^4 - h_{6,1}B_4^3 + h_{4,2}B_4^2 - h_{2,3}B_4 + 1 = 0. \end{cases} \qquad (42)$$

Using the explicit values of B_k from (20)–(23), we obtain that

$$h_{8,0} = -\frac{1}{3}, \quad h_{4,2} = 0, \quad h_{6,1} = 0, \quad h_{2,3} = 0. \qquad (43)$$

From other equations of the algebraic system we obtain

$$h_{6,0} = \frac{4}{3}g_r, \quad h_{4,0} = -\frac{16}{9}g_r^2, \quad h_{2,0} = \frac{64}{81}g_r^3, \qquad (44)$$

all other $h_{i,j}$ are equal to zero. So, the equation for M is

$$M'^4 = \frac{1}{81} M^2 \left(3M^2 - 4g_r\right)^3. \qquad (45)$$

Eq. (14) at $u_i = 1$, $u_r = \tilde{c} = d_r = d_i = g_i = 0$ has the form:

$$\frac{1}{4}\left(M'''\right)^2 - \left(2MM'' - M'^2\right)\left(M^2 - g_r\right)^2 = 0. \qquad (46)$$

A straightforward calculation shows that any solution of (45) satisfies (46). So, we obtain elliptic wave solutions of the CGLE5.

Summing up we can conclude that our modification of the Conte–Musette method allows us to get two results: we obtain new elliptic wave solutions of the CGLE5, and we prove that these solutions are unique elliptic solutions for the CGLE5 with $g_r \neq 0$.

3 Construct of elliptic solutions for nonintegrable systems

We propose the following way to the search for elliptic solutions of nonintegrable systems:

1. Calculate a few first terms of all solutions of system (1) in the form of the Laurent series.
2. Choose the function y_k, which should be elliptic. Check should other functions be elliptic or not.
3. Using the residue theorem define values of numeric parameters at which the solution y_k can be an elliptic function.
4. Define a minimal number of poles for candidates to elliptic solutions.

5. Calculate the sufficient number of coefficients for all Laurent series of y_k and substitute the obtained coefficients into Eq. (2). This substitution transforms (2) into a linear and overdetermined system in $h_{j,k}$ with coefficients depending on parameters.
6. Eliminate coefficients $h_{j,k}$ and get a system in parameters.
7. Solve the obtained nonlinear system.

4 Conclusions:

- We propose a new approach for the search of elliptic solutions to systems of differential equations. The proposed algorithm is a modification of the Conte–Musette method. We restrict ourselves to the search of elliptic solutions only.

- A key idea of this restriction is to simplify calculations by means of the use of a few Laurent series solutions instead of one and the use of the residue theorem.
- The application of our approach to the quintic complex one-dimensional Ginzburg–Landau equation (CGLE5) allows to find elliptic solutions in the wave form. Note that the obtained solutions are the first elliptic solutions for the CGLE5.
- Using the investigation of the CGLE5 as an example, we demonstrate that to find elliptic solutions the analysis of a system of differential equations is more preferable than the analysis of the equivalent single differential equation.
- We also find restrictions on coefficients, which are necessary conditions for the existence of elliptic solutions for the CGLE5.