# Operator algebras related to bounded positive operator with simple spectrum

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# Coherent state map and corresponding algebra

$\mathbf{H}$ – a bounded positive operator with simple spectrum in Hilbert space $\mathcal{H}$

$|0\rangle$ – cyclic vector, i.e. $\{E(\Delta)\,|0\rangle\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ is linearly dense in $\mathcal{H}$

$$\langle 0|0\rangle = 1$$

$$I : L^2(\mathbb{R}, d\mu) \ni f \longmapsto \int_{\mathbb{R}} f(\lambda)E(d\lambda)\,|0\rangle \in \mathcal{H}$$

is an isomorphism for

$$\mu(\Delta) = \langle 0|E(\Delta)\,0\rangle.$$

$I^* \circ \mathbf{H} \circ I$ acts in $L^2(\mathbb{R}, d\mu)$ as the multiplication by argument

By Gram-Schmidt orthonormalization one obtains orthonormal polynomials $P_n$ in $L^2(\mathbb{R}, d\mu)$

The orthonormal basis in $\mathcal{H}$

$$|n\rangle := P_n(\mathbf{H})\,|0\rangle = I(P_n)$$

$$\mathbf{H}\,|n\rangle = b_{n-1}\,|n-1\rangle + a_n\,|n\rangle + b_n\,|n+1\rangle$$

$b_n > 0$, $b_{-1} = 0$ and $a_n, b_n \in \mathbb{R}$

Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & \cdots \\ b_0 & a_1 & b_1 & \\ 0 & b_1 & a_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

**Moments** of measure $\mu$

$$\sigma_k := \int_{\mathbb{R}} \lambda^k \mu(d\lambda) > 0$$

$$\sigma_k = \langle 0|\mathbf{H}^k\, 0\rangle, \qquad k \in \mathbb{N} \cup \{0\}$$

Resolvent $R_\lambda := (\mathbf{H} - \lambda\mathbb{1})^{-1}$

$$\langle 0|R_\lambda\, 0\rangle = \int_{\mathbb{R}} \frac{\mu(dx)}{x-\lambda} = -\sum_{k=0}^{\infty} \frac{\sigma_k}{\lambda^{k+1}},$$

where the second equality is valid for $|\lambda| > \|\mathbf{H}\|$.

**Coherent state map** – a complex analytic map $K : \mathbb{D} \to \mathcal{H}$ from disc $\mathbb{D}$ with linearly dense image, expressed by

$$K(z) = \sum_{n=0}^{\infty} c_n z^n |n\rangle , \qquad 0 < c_n \in \mathbb{R}$$

**Annihilation operator**

$$\mathbf{A}K(z) = zK(z).$$

$$\mathbf{A} |n\rangle := \frac{c_{n-1}}{c_n} |n-1\rangle ,$$

where $c_{-1} = 0$.
Creation operator

$$\mathbf{A}^* |n\rangle = \frac{c_n}{c_{n+1}} |n+1\rangle .$$

Generalized exponential function $E : \tilde{\mathbb{D}} \to \mathbb{C}$

$$E(\bar{v}w) := \langle K(v)|K(w)\rangle,$$

We consider two coherent state maps

$$K_1(z) := \sum_{n=0}^{\infty} \sqrt{\sigma_n} z^n |n\rangle ,$$

for $|z| < \|\mathbf{H}\|^{-\frac{1}{2}}$

$$K_2(z) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{\sigma_n}} z^n |n\rangle$$

for $|z| < \|\mathbf{H}\|^{\frac{1}{2}}$.

$$E_1(z) = -\frac{1}{z} \langle 0| R_{\frac{1}{z}} |0\rangle = \sum_{n=0}^{\infty} \sigma_n z^n$$

Decomposition of unity

$$\int |K_2(z)\rangle\langle K_2(z)| \, \nu(dz) = \mathbb{1},$$

$$\nu(dz) := \frac{1}{2\pi} d\varphi \; f^*\mu(dr),$$

where $z = re^{i\varphi}$ and $f^*\mu$ is pullback of (2) by $f(x) := x^2$.

$$E_2(\overline{v}w) = \int E_2(\overline{v}z) E_2(\overline{z}w) \, \nu(dz),$$

$\mathcal{T}$ - Toeplitz algebra, i.e. $C^*$-algebra generated by shift operator

$$\mathbf{S}\,|n\rangle = |n-1\rangle\,, \quad n \in \mathbb{N}$$
$$\mathbf{S}\,|0\rangle = 0.$$

## Proposition

i) *Let us assume that $\mathbf{A}_1$ is bounded. Then the $C^*$-algebra $\mathcal{A}_1$ generated by $\mathbf{A}_1$ coincides with $\mathcal{T}$ if and only if the sequence $\{\frac{\sigma_{n-1}}{\sigma_n}\}_{n\in\mathbb{N}}$ is convergent.*

ii) *Let us assume that $\mathbf{A}_2$ is bounded. Then the $C^*$-algebra $\mathcal{A}_2$ generated by $\mathbf{A}_2$ coincides with $\mathcal{T}$ if and only if the sequence $\{\frac{\sigma_{n-1}}{\sigma_n}\}_{n\in\mathbb{N}}$ is convergent.*

iii) *If both $\mathbf{A}_1$ and $\mathbf{A}_2$ are bounded then $\mathcal{A}_1 = \mathcal{A}_2$.*

## Sketch of proof

i)

$$\liminf_{n\to\infty} \frac{\sigma_n}{\sigma_{n-1}} \leqslant \liminf_{n\to\infty} \sqrt[n]{\sigma_n} \leqslant \limsup_{n\to\infty} \sqrt[n]{\sigma_n} \leqslant \limsup_{n\to\infty} \frac{\sigma_n}{\sigma_{n-1}}$$

$$\lim_{n\to\infty} \sqrt[n]{\sigma_n} = \lim_{n\to\infty} \frac{\sigma_n}{\sigma_{n-1}}$$

$$\mathbf{S} = (\mathbf{A}_1\mathbf{A}_1^*)^{-\frac{1}{2}}\mathbf{A}_1 \in \mathcal{A}_1$$

$\mathbf{A}_1\mathbf{A}_1^* - (\|\mathcal{H}\|)^{-1}\mathbb{1}$ is compact. Thus $\mathbf{A}_1 = (\mathbf{A}_1\mathbf{A}_1^*)^{\frac{1}{2}}\mathbf{S} \in \mathcal{T}$.

iii) $\mathbf{A}_1$ bounded $\Rightarrow$ $\mathbf{A}_2\mathbf{A}_2^*$ bounded from below

$$\mathbf{A}_1\mathbf{A}_1^* = (\mathbf{A}_2\mathbf{A}_2^*)^{-1}$$

$$\mathbf{A}_1 = (\mathbf{A}_2\mathbf{A}_2^*)^{-1}\mathbf{A}_2$$

$$\mathbf{A}_2 = (\mathbf{A}_1\mathbf{A}_1^*)^{-1}\mathbf{A}_1$$

We will restrict our considerations to the case $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{T}$.

$$\mathbf{Q} := \sum_{n=0}^{\infty} q^n |n\rangle\langle n| \in \mathcal{T}, \qquad 0 < q < 1$$

Structural function $\mathcal{R} : \operatorname{spec}\mathbf{Q} \to \operatorname{spec}\mathbf{A}_1^*\mathbf{A}_1$ (continuous)

$$\mathcal{R}(q^n) := \frac{\sigma_{n-1}}{\sigma_n} \quad \text{for} \quad n \in \mathbb{N} \cup \{0\},$$

$$\mathcal{R}(0) := \lim_{n\to\infty} \frac{\sigma_{n-1}}{\sigma_n} = \|\mathbf{H}\|^{-1},$$

Structural relations:

$$\mathbf{A}_1^*\mathbf{A}_1 = \mathcal{R}(\mathbf{Q})$$

$$\mathbf{A}_1\mathbf{A}_1^* = \mathcal{R}(q\mathbf{Q})$$

$$q\mathbf{Q}\mathbf{A}_1 = \mathbf{A}_1\mathbf{Q}$$

$$q\mathbf{A}_1^*\mathbf{Q} = \mathbf{Q}\mathbf{A}_1^*$$

$$\mathbf{N}\,|n\rangle := n\,|n\rangle$$

$$[\mathbf{A}_1, \mathbf{N}] = \mathbf{A}_1$$

$$[\mathbf{A}_1^*, \mathbf{N}] = -\mathbf{A}_1^*$$

$$\mathbf{A}_1^{*k}\mathbf{A}_1^k = \mathcal{R}(\mathbf{Q})...\mathcal{R}(q^k\mathbf{Q}), \quad \mathbf{A}_1^k\mathbf{A}_1^{*k} = \mathcal{R}(q\mathbf{Q})...\mathcal{R}(q^{k+1}\mathbf{Q})$$

$$\sigma_k = \frac{1}{\mathcal{R}(q)...\mathcal{R}(q^k)} = \frac{1}{\langle 0|\mathbf{A}_1^k\mathbf{A}_1^{*k}\,0\rangle}$$

$$E_1(z) := \sum_{n=0}^{\infty} \frac{z^n}{\mathcal{R}(q)...\mathcal{R}(q^k)},$$

$$E_2(z) := \sum_{n=0}^{\infty} \mathcal{R}(q)...\mathcal{R}(q^k)\; z^n.$$

## Example (little $q$-Jacobi polynomials)

$$\mathcal{R}(x) = \frac{(1-x)(1-b_1q^{-1}x)\dots(1-b_{r-1}q^{-1}x)}{(1-a_1q^{-1}x)\dots(1-a_rq^{-1}x)}(1-\chi_{\{1\}}(x)),$$

$\chi_{\{1\}}$ is a characteristic function of the set $\{1\}$
$a_i, b_i < 1$.
Generalized exponential functions — basic hypergeometric series

$$E_1(z) = {}_r\Phi_{r-1}\left(\begin{matrix} a_1 \dots \ a_r \\ b_1 \dots b_{r-1} \end{matrix} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \dots (a_r;q)_n}{(q;q)_n(b_1;q)_n \dots (b_{r-1};q)_n} z^n$$

$$E_2(z) = {}_{r+1}\Phi_r\left(\begin{matrix} q \ q \ \ b_1 \dots b_{r-1} \\ a_1 \dots a_r \end{matrix} \middle| q; z\right)$$

for $|z| < 1$

## Example (little $q$-Jacobi polynomials)

Moments

$$\sigma_n = \frac{(a_1;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_{r-1};q)_n},$$

$q$-Pochhammer symbol

$$(\alpha;q)_n := (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1})$$

## Example (little $q$-Jacobi polynomials, case $r = 1$)

$$\mathcal{R}(x) = \frac{1-x}{1-ax}$$

$1 < a < q^{-1}$

Structural relations

$$1 - \mathbf{Q} = (1 - a\mathbf{Q})\mathbf{A}_1^*\mathbf{A}_1$$

$$1 - q\mathbf{Q} = (1 - aq\mathbf{Q})\mathbf{A}_1\mathbf{A}_1^*$$

Moments are

$$\sigma_k = \frac{(aq;q)_k}{(q;q)_k}$$

## Example (little $q$-Jacobi polynomials, case $r = 1$)

Coefficients of Jacobi matrix

$$a_n = \frac{q^n(1-aq^{n+1})(1-q^n)}{(1-q^{2n})(1-q^{2n+1})} + \frac{aq^n(1-q^n)(1-a^{-1}q^{n-1})}{(1-q^{2n-1})(1-q^{2n})},$$

$$b_n = \sqrt{\frac{aq^{2n+1}(1-q^{n+1})(1-aq^{n+1})(1-a^{-1}q^n)(1-q^n)}{(1-q^{2n})(1-q^{2n+1})^2(1-q^{2n+2})}}.$$

Measure $\mu$ is discrete

$$\mu(d\lambda) = \frac{(aq;q)_\infty(a^{-1},q)_\infty}{(q;q)_\infty(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q\lambda;q)_n a^n \lambda}{(a^{-1}\lambda;q)_n} \delta(\lambda - q^n) d\lambda.$$

The polynomials orthonormal with respect to this measure are a subclass of little $q$-Jacobi polynomials.

## Example (little $q$-Jacobi polynomials, case $r=1$)

Exponential functions

$$E_1(z) = {}_1\Phi_0 \left( \begin{matrix} aq \\ - \end{matrix} \,\middle|\, q; z \right)$$

$$E_2(z) = {}_2\Phi_1 \left( \begin{matrix} q & q \\ a & \end{matrix} \,\middle|\, q; z \right)$$

Reproducing property

$$E_2(\overline{v}w) = \frac{1}{2\pi}\int_0^1 dr \int_0^{2\pi} d\varphi E_2(\overline{v}re^{i\varphi})E_2(re^{-i\varphi}w)\times$$

$$\frac{(aq;q)_\infty(a^{-1},q)_\infty}{(q;q)_\infty(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(qr;q)_n a^n r}{(a^{-1}r;q)_n}\delta(r-q^{2n})$$

## Example (little $q$-Jacobi polynomials, case $r = 2$ and $a_1 = q$, $b_1 < a_2$, $0 < a_2 < 1$)

$$\mathcal{R}(x) = \frac{(1 - b_1 q^{-1} x)}{(1 - a_2 q^{-1} x)}(1 - \chi_{\{1\}}(x))$$

Structural relations

$$(1 - a_2 q^{-1}\mathbf{Q})\mathbf{A}_1^*\mathbf{A}_1 = (1 - b_1 q^{-1}\mathbf{Q})(1 - |0\rangle\langle 0|)$$

$$(1 - a_2\mathbf{Q})\mathbf{A}_1\mathbf{A}_1^* = 1 - b_1\mathbf{Q},$$

Moments $\sigma_n$

$$\sigma_n = \frac{(a_2; q)_n}{(b_1; q)_n}$$

## Example (little $q$-Jacobi polynomials, case $r = 2$ and $a_1 = q$, $b_1 < a_2$, $0 < a_2 < 1$)

Coefficients of Jacobi matrix

$$a_n = \frac{q^n(1-a_2q^n)(1-b_1q^{n-1})}{(1-b_1q^{2n-1})(1-b_1q^{2n})} + \frac{a_2q^{n-1}(1-q^n)(1-q^{n-1}b_1/a_2)}{(1-b_1q^{2n-2})(1-b_1q^{2n-1})}$$

$$b_n = \sqrt{\frac{a_2q^{2n}(1-q^{n+1})(1-q^nb_1/a_2)(1-a_2q^n)(1-b_1q^{n-1})}{(1-b_1q^{2n-1})(1-b_1q^{2n})^2(1-b_1q^{2n+1})}}$$

Measure

$$\mu(d\lambda) = \frac{(a_2;q)_\infty(b_1/a_2;q)_\infty}{(b_1;q)_\infty(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q\lambda;q)_\infty a_2^n}{(\lambda b_1/a_2;q)_\infty}\delta(\lambda-q^n)d\lambda$$

## Example (little $q$-Jacobi polynomials, case $r = 2$ and $a_1 = q$, $b_1 < a_2$, $0 < a_2 < 1$)

Exponential functions

$$E_1(z) = {}_2\Phi_1 \left( \begin{matrix} a_2 & q \\ b_1 & \end{matrix} \middle| q; z \right)$$

$$E_2(z) = {}_2\Phi_1 \left( \begin{matrix} b_1 & q \\ a_2 & \end{matrix} \middle| q; z \right)$$

Reproducing property

$$E_2(\overline{v}w) = \frac{1}{2\pi} \int_0^1 dr \int_0^{2\pi} d\varphi E_2(\overline{v}re^{i\varphi}) E_2(re^{-i\varphi}w) \times$$

$$\times \frac{(a_2;q)_\infty (b_1/a_2;q)_\infty}{(b_1;q)_\infty (q;q)_\infty} \sum_{n=0}^{\infty} \frac{(rq;q)_\infty a_2^n}{(rb_1/a_2;q)_\infty} \delta(r - q^{2n})$$

**Example (little $q$-Jacobi polynomials, case $r = 2$ and $a_1 = q$, $b_1 < a_2$, $0 < a_2 < 1$)**

Orthogonal polynomials corresponding to this case are the little $q$-Jacobi polynomials.

- $b_1 = q$ — previous case $r = 1$
- $b_1 = q^2$, $a_2 = q$ — the little $q$-Legendre polynomials
- $b_1 = 0$, $0 < a_2 < 1$ — the little $q$-Laguerre/Wall polynomials

## Example (Classical Jacobi polynomials)

Coefficients of Jacobi matrix $(\alpha, \beta > -1)$

$$a_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}$$

$$b_n = 2\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+1+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}}$$

Measure

$$\mu(d\lambda) = (1-\lambda)^\alpha \lambda^\beta \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \chi_{[0,1]}(\lambda) d\lambda$$

Moments

$$\sigma_n = \frac{(\beta+1)_n}{(\alpha+\beta+2)_n}$$

## Example (Classical Jacobi polynomials)

Structural function

$$R(q^x) = \frac{\alpha + \beta + 1 + x}{\beta + x}(1 - \chi_{\{1\}}(x))$$

Structural relations

$$(\beta + \mathbf{N})\mathbf{A}_1^*\mathbf{A}_1 = (\alpha + \beta + 1 + \mathbf{N})(1 - |0\rangle\langle 0|)$$

$$(\beta + 1 + \mathbf{N})\mathbf{A}_1\mathbf{A}_1^* = \alpha + \beta + 2 + \mathbf{N}$$

## Example (Classical Jacobi polynomials)

Exponential functions

$$E_1(z) = {}_2F_1 \left( \begin{matrix} \beta+1 & 1 \\ \alpha+\beta+2 & \end{matrix} \middle| z \right)$$

$$E_2(z) = {}_2F_1 \left( \begin{matrix} \alpha+\beta+2 & 1 \\ \beta+1 & \end{matrix} \middle| z \right)$$

Reproduction property of $E_2$ holds for measure

$$\nu(dz) = \frac{1}{2\pi}(1-r^2)^\alpha r^{2\beta} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \chi_{[0,1]}(r) d\varphi dr$$

## Example (Classical Jacobi polynomials)

Subcases:

- $\alpha = \beta = \lambda - \frac{1}{2}$ — Gegenbauer/ultraspherical polynomials
- $\alpha = \beta = -\frac{1}{2}$ – Chebychev I kind
- $\alpha = \beta = \frac{1}{2}$ – Chebychev II kind
- $\alpha = \beta = 0$ – Legendre/spherical

# Toda isospectral deformation

One-parameter subgroup

$$\mathbb{R} \ni t \longmapsto \mathbf{U}_t \in \mathsf{Aut}\,\mathcal{H}$$

such that matrix $J$

$$\mathbf{H} = \sum_{n,m=0}^{\infty} (J_t)_{nm} \left|m\right\rangle_t {}_t\left\langle n\right|$$

is three-diagonal

$$\left|n\right\rangle_t := \mathbf{U}_t \left|n\right\rangle .$$

Evolution of basis

$$\frac{d}{dt} |n\rangle_t = \mathbf{B}_t^* |n\rangle_t ,$$

$$\mathbf{B}_t := \left( \frac{d}{dt} \mathbf{U}_t \right) \mathbf{U}_t^*.$$

Instead of considering the evolution of the basis, one can consider the evolution of $\mathbf{H}$

$$\mathbf{H}_t := \mathbf{U}_t^* \mathbf{H} \mathbf{U}_t = \sum_{n,m=0}^{\infty} (J_t)_{nm} \left| m \right\rangle \left\langle n \right|$$

$$\frac{d}{dt} \mathbf{H}_t = [\mathbf{H}_t, \mathbf{B}_t]$$

with condition that $\left| 0 \right\rangle$ is cyclic for all $\mathbf{H}_t$ and $J_t$ is three-diagonal

Let $\mathbf{H}_t$ depend on infinite number of "times" $t = (t_1, t_2, \ldots)$

Toda lattice equations

$$\frac{\partial}{\partial t_k}\mathbf{H}_t = [\mathbf{H}_t, \mathbf{B}_{k\,t}]$$

$$\mathbf{B}_{k\,t} := \mathbf{H}_t^k - P_0(\mathbf{H}_t^k) - 2P_+(\mathbf{H}_t^k)$$

$P_0(\mathbf{H}_t^k)$ is diagonal operator

$$P_0(\mathbf{H}_t^k) := \sum_{n=0}^{\infty}(J_t^k)_{nn}|n\rangle\langle n|$$

$P_+(\mathbf{H}_t^k)$ is upper-triangular operator

$$P_+(\mathbf{H}_t^k) := \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(J_t^k)_{mn}|m\rangle\langle n|$$

Moments $\sigma_k(t)$ satisfy the equations

$$\frac{\partial}{\partial t_l}\sigma_k(t) = 2(\sigma_{k+l}(t) - \sigma_k(t)\sigma_l(t))$$

Since $\frac{\partial}{\partial t_k}\sigma_l(t) = \frac{\partial}{\partial t_l}\sigma_k(t)$ then there exists a function $\tau(t) = \tau(t_1, t_2, \ldots)$ such that

$$\sigma_k(t) = \frac{1}{2}\frac{\partial}{\partial t_k}\log\tau(t)$$

Evolution equation in terms of $\tau$

$$\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_l}\tau(t) = 2\frac{\partial}{\partial t_{k+l}}\tau(t)$$

Equation on measure $\mu_t$

$$\frac{\partial}{\partial t_k}\mu_t(d\lambda) = \left(\lambda^k - \int_{\mathbb{R}} \gamma^k \mu_t(d\gamma)\right)\mu_t(d\lambda)$$

Solution of evolution equations

$$\tau(t) = \tau(0)\int_{\mathbb{R}} e^{2\sum_{l=1}^{\infty} t_l\lambda^l}\mu_0(d\lambda)$$

$$\mu_t(d\lambda) = \frac{e^{2\sum_{l=1}^{\infty} t_l\lambda^l}}{\int_{\mathbb{R}} e^{2\sum_{l=1}^{\infty} t_l\gamma^l}\mu_0(d\gamma)}\mu_0(d\lambda)$$

$$\sigma_k(t) = \frac{1}{\int_{\mathbb{R}} e^{2\sum_{l=1}^{\infty} t_l\gamma^l}\mu_0(d\gamma)}\int_{\mathbb{R}} \lambda^k e^{2\sum_{l=1}^{\infty} t_l\lambda^l}\mu_0(d\lambda),$$

Evolution equation on structural function

$$\frac{\partial}{\partial t_l}\mathcal{R}_t(\mathbf{Q}) = 2\mathcal{R}_t(\mathbf{Q})\left(\frac{1}{\mathcal{R}_t(\mathbf{Q})\mathcal{R}_t(q\mathbf{Q})\dots\mathcal{R}_t(q^{l-1}\mathbf{Q})} - \right.$$

$$\left. - \frac{1}{\mathcal{R}_t(q\mathbf{Q})\mathcal{R}_t(q^2\mathbf{Q})\dots\mathcal{R}_t(q^{l}\mathbf{Q})}\right)$$

Hierarchy of equations on annihilation and creation operators

$$\frac{\partial}{\partial t_l}\mathbf{A}_{1\,t} = [(\mathbf{A}_{1\,t}^{l}\mathbf{A}_{1\,t}^{*l})^{-1}, \mathbf{A}_{1\,t}] \qquad \frac{\partial}{\partial t_l}\mathbf{A}_{1\,t}^{*} = -[(\mathbf{A}_{1\,t}^{l}\mathbf{A}_{1\,t}^{*l})^{-1}, \mathbf{A}_{1\,t}^{*}]$$

$$\frac{\partial}{\partial t_l}\mathbf{A}_{2\,t} = -[(\mathbf{A}_{2\,t}^{l}\mathbf{A}_{2\,t}^{*l})^{-1}, \mathbf{A}_{2\,t}] \qquad \frac{\partial}{\partial t_l}\mathbf{A}_{2\,t}^{*} = [(\mathbf{A}_{2\,t}^{l}\mathbf{A}_{2\,t}^{*l})^{-1}, \mathbf{A}_{2\,t}^{*}]$$

**Proposition**

*Provided that* $\lim \frac{\sigma_{n-1}(0)}{\sigma_n(0)}$ *exists,* $\lim \frac{\sigma_{n-1}(t)}{\sigma_n(t)}$ *exists for all* $t$*.*

Thus if $\mathbf{A_1} = \mathcal{T}$ for $t = 0$ it is also true for any $t$.

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