An Introduction to the Dirac Operator in Riemannian Geometry

S. Montiel

Universidad de Granada

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Genealogy of the Dirac Operator

- 1913 É. Cartan Orthogonal Lie algebras
- 1927 W. Pauli Inner angular momentum (*spin*) of electrons
- 1928 P.A.M. Dirac Dirac operator and quantum-relativistic description of electrons
- 1930 H. Weyl Wave functions of neutrinos
- 1937 É. Cartan *Insurmountables difficulties* to talk about spinors on manifolds
- 1963 M. Atiyah and I. Singer Dirac operator on a spin Riemannian manifold

Genealogy of the Dirac Operator

- 1963 A. Lichnerowicz (maybe I. Singer in the last 50's) Topological obstruction for positive scalar curvature on compact spin manifolds
- 1974 N. Hitchin The dimension of the space of harmonic spinors is a conformal invariant and existence of parallel spinors implies special holonomy
- 1980 M. Gromov and B. Lawson More topological obstructions for complete metrics with non-negative scalar curvature
- 1981 E. Witten An *elemental* spinorial proof of the Schoen and Yau positive mass theorem
- 1995 E. Witten Seiberg–Witten $\Rightarrow$ Donaldson

The Wave Equation (1850-1905)

- Wave equation of Maxwell and Special Relativity theories

$$O \subset \mathbb{R}^3 \qquad u : O \times \mathbb{R} \longrightarrow \mathbb{R} \qquad \Box u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \Delta u = 0$$

$$u(p,t) = \sum f(t)\phi(p) \qquad f'' + \lambda f = 0 \qquad \Delta\phi - \lambda\phi = 0$$

where $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$ and c is the ratio between electrostatic and electrodynamic units of charge

- Second order in time and space coordinates

- Invariant under Lorentz transformations

$$O(1,3) = \{A \in GL(4,\mathbb{R}) \,|\, AGA^t = G\}, \qquad G = \mathrm{diag}\,(-1,1,1,1)$$

The Schrödinger Equation (1926)

- Schrödinger equation of the non-relativistic Quantum Mechanics

$$O \subset \mathbb{R}^3 \qquad \psi : O \times \mathbb{R} \longrightarrow \mathbb{C} \qquad -i\frac{\partial \psi}{\partial t} + \Delta\psi = 0$$

$$\psi(p,t) = \sum f(t)\phi(p) \qquad f' + i\lambda f = 0 \qquad \Delta\phi - \lambda\phi = 0$$

- Invariant under Galileo transformations

$$\mathbb{R}^3 \cdot O(3) = \{A \in GL(4,\mathbb{R}) \,|\, A = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}, v \in \mathbb{R}^3, A \in O(3)\}$$

- First order in time and second order in space coordinates

- Complex values

The Classical Dirac Operator

- 1928 P.A.M. Dirac, 1930 H. Weyl

Look for an equation of first order in all the variables, like this

$$\frac{i}{c}\frac{\partial \psi}{\partial t} + D\psi = 0 \qquad D\psi = \sum_{i=1}^{3} \gamma_i \frac{\partial \psi}{\partial x_i}$$

whose iteration on solutions gives the wave equation. This holds iff

$$D^2 = \Delta \iff \gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}$$

for $i, j = 1, 2, 3$. For example, these Pauli matrices

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The Classical Dirac Operator and Spinor Fields

- Other possible Pauli matrices $\gamma_i' = P\gamma_i P^{-1}$ or $\gamma_i' = -\gamma_i$.
- Two essentially different (chirality) Dirac-Weyl equations

$$\pm\frac{i}{c}\frac{\partial\psi}{\partial t} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\frac{\partial\psi}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\partial\psi}{\partial y} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\frac{\partial\psi}{\partial z} = 0$$

- Spinor fields $\psi : O \times \mathbb{R} \longrightarrow \mathbb{C}^2$ expand into series

$$\psi(p,t) = \sum f(t)\phi(p) \qquad f' + i\lambda f = 0$$

$$D\phi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\frac{\partial\phi}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\partial\phi}{\partial y} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\frac{\partial\phi}{\partial z} = -\lambda\phi$$

The Classical Dirac Operator

• To define the Dirac operator in terms of any other orthonormal basis $\{e_1, e_2, e_3\}$ as

$$D = \gamma(e_1)\nabla_{e_1} + \gamma(e_2)\nabla_{e_2} + \gamma(e_3)\nabla_{e_3}$$

we need Pauli matrices for all directions $v \in \mathbb{R}^3$. Put

$$\gamma(v) = \gamma(v_1, v_2, v_3) = v_1\gamma_1 + v_2\gamma_2 + v_3\gamma_3 = \begin{pmatrix} iv_1 & v_2 + iv_3 \\ -v_2 + iv_3 & -iv_1 \end{pmatrix}$$

• A Lie algebra isomorphism

$$\gamma : (\mathbb{R}^3 = \mathfrak{o}(3), \wedge) \to (\mathfrak{su}(2), \frac{1}{2}[\ ,\ ])$$

• Clifford relations

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2\langle u, v\rangle I_2, \qquad \forall u, v \in \mathbb{R}^3$$

What Are These Spinor Fields?

• For each $A \in SU(2)$ there is a unique real matrix $\rho(A) \in M_{\mathbb{R}}(3)$ such that

$$\gamma(\rho(A)v) = A\gamma(v)\bar{A}^t \qquad \forall v \in \mathbb{R}^3$$

• See that $\rho(A) \in SO(3)$ and the map $\rho : SU(2) \to SO(3)$ is a two-sheeted (universal) covering group homomorphism

• Surjective: given $R \in SO(3)$, put $R = s_1 \circ s_2$ and prove that $A = \gamma(v_1)\gamma(v_2) \in SU(2)$ and that $\rho(A) = R$

• Kernel: if $A \in \ker \rho$ then A commutes with all Pauli matrices and so $A = \pm I_2$

What Are These Spinor Fields?

• Let $\phi : O \to \mathbb{C}^2$ be a spinor and $A \in SU(2)$. Consider the open set $O' = \rho(A)^t(O)$ and define

$$\psi : O' \to \mathbb{C}^2, \qquad \psi(p) = \bar{A}^t \phi(\rho(A)p), \qquad \forall p \in O'$$

$$(D\psi)(p) = \sum_{i=1}^3 \gamma(e_i)(\nabla_{e_i}\psi)(p) = \sum_{i=1}^3 \gamma(e_i)\bar{A}^t(\nabla_{\rho(A)e_i}\phi)(\rho(A)p)$$
$$= \sum_{i=1}^3 \bar{A}^t\gamma(\rho(A)e_i)(\nabla_{\rho(A)e_i}\phi)(\rho(A)p) = \bar{A}^t(D\phi)(\rho(A)p)$$

and so $D\phi = \lambda\phi \Leftrightarrow D\psi = \lambda\psi$

• If spatial coordinates change through $R \in SO(3)$, then components of spinors change through $\rho^{-1}(R) \in SU(2)$(?)

• $\rho \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} = R_\theta$ is a rotation of angle θ around the x-axis

Speaking Bundle Language

- 1963 Atiyah and Singer

⋆ $O \subset \mathbb{R}^3$ is a chart domain of an oriented Riemannian three-manifold M

⋆ $\phi : O \to \mathbb{C}^2$ is the local expression of a section of a complex vector bundle ΣM with fiber $\mathbb{C}^2$ associated to a virtual (?) principal bundle with structure group $SU(2)$

⋆ Lift transition functions $f_{ij} : U_i \cap U_j \to SO(3)$ to maps $g_{ij} : U_i \cap U_j \to SU(2)$ and define

$$h_{ijk} : U_i \cap U_j \cap U_k \to \mathbb{Z}_2 = \{+1, -1\}$$

according to $g_{ik} = \pm(g_{jk} g_{ij})$. This h is a cocycle and defines the *second Stiefel-Whitney class* $w_2(M) \in H^2(M, \mathbb{Z}_2)$

Speaking Bundle Language

⋆ ΣM must have a Hermitian metric $\langle\ ,\ \rangle$ and a covariant derivative ∇ which parallelizes the metric

⋆ A bundle map $\gamma : TM \to \mathrm{End}_{\mathbb{C}}(\Sigma M)$ with

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2\langle u, v\rangle I_2$$

compatible with both $\langle\ ,\ \rangle$ and ∇ called a *Clifford multiplication* because it determines a complex representation of each Clifford algebra $\mathbb{C}\ell(T_pM)$ on the space $\Sigma_p M$

⋆ In this frame, the Dirac operator is

$$D\psi = \sum_{i=1}^{3} \gamma(e_i)\nabla_{e_i}\psi$$

where e_1, e_2, e_3 is an orthonormal basis of T_pM

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Emergence Kit for Riemannian Geometry Notation

- Let M be a Riemannian manifold, $\langle\ ,\ \rangle$ the metric and ∇ the Levi-Cività connection

- R will be the Riemannian curvature operator

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X,Y,Z \in \Gamma(TM)$$

and also Riemannian curvature tensor of M, given by

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle, \qquad X,Y,Z,W \in \Gamma(TM)$$

- The single and double contractions of this four-covariant tensor

$$\mathrm{Ric}(X,W) = \sum_{i=1}^{n} R(X,e_i,e_i,W) \qquad S = \sum_{i,j=1}^{n} R(e_i,e_j,e_j,e_i)$$

are the Ricci tensor and the scalar curvature of M, respectively

Exterior Geometry for Riemannian Geometers

- The exterior bundle $\Lambda^*(M) = \bigoplus_{k=1}^n \Lambda^k(M)$ inherits the metric and the connection
- Lemma 1 Music and products are parallel

$$\nabla_X(Y^\flat) = (\nabla_X Y)^\flat, \qquad \nabla_X(\alpha^\sharp) = (\nabla_X \alpha)^\sharp$$

$$\nabla_X(\omega \wedge \eta) = (\nabla_X \omega) \wedge \eta + \omega \wedge (\nabla_X \eta)$$

$$\nabla_X(Y \lrcorner\, \omega) = (\nabla_X Y) \lrcorner\, \omega + Y \lrcorner\, (\nabla_X \omega)$$

- Exterior product and inner product are adjoint each other

$$\langle X^\flat \wedge \omega, \eta \rangle = \langle \omega, X \lrcorner\, \eta \rangle$$

- Riemannian expressions for an old friend and its adjoint

$$d = \sum_{i=1}^n e_i^\flat \wedge \nabla_{e_i} \qquad \delta = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i}$$

Hermitian Bundles

- Let ΣM be a rank N complex vector bundle over M, $\langle\ ,\ \rangle$ a Hermitian metric and ∇ a unitary connection with

$$X\langle\psi,\phi\rangle = \langle\nabla_X\psi,\phi\rangle + \langle\psi,\nabla_X\phi\rangle \qquad \psi,\phi\in\Gamma(\Sigma M), X\in\Gamma(TM)$$

- The Levi-Cività connection allows us to perform second derivatives

$$(\nabla^2\psi)(X,Y) = \nabla_X\nabla_Y\psi - \nabla_{(\nabla_X Y)}\psi$$

- The skew-symmetric part

$$R^{\Sigma M}(X,Y)\psi = (\nabla^2\psi)(X,Y) - (\nabla^2\psi)(Y,X)$$

is tensorial in ψ. It is the *curvature operator* of $(\Sigma M,\nabla)$

- Skew-symmetry and Bianchi identity

$$\langle R^{\Sigma M}(X,Y)\psi,\phi\rangle = -\langle\psi,R^{\Sigma M}(X,Y)\phi\rangle$$
$$(\nabla_Z R^{\Sigma M})(X,Y) + (\nabla_Y R^{\Sigma M})(Z,X) + (\nabla_X R^{\Sigma M})(Y,Z) = 0$$

Hermitian Bundles

- As a consequence

$$\alpha(X,Y) = \mathrm{tr}\, iR^{\Sigma M}(X,Y) = -\sum_{k=1}^{N} \langle R^{\Sigma M}(X,Y)\psi_k, i\psi_k \rangle$$

is a closed two-form with $2\pi\mathbb{Z}$-periods

- The first Chern class

$$c_1(\Sigma M) = [\frac{1}{2\pi}\alpha] \in H^2(M,\mathbb{Z})$$

does not depend on the connection ∇

- When $N = 1$ (complex line bundles case)

$$c_1 : (H^1(M,\mathbb{S}^1),\otimes) \to (H^2(M,\mathbb{Z}),+)$$

is an isomorphism

The Rough Laplacian

- Second derivatives allow to define the *rough Laplacian*

$$\Delta : \Gamma(\Sigma M) \to \Gamma(\Sigma M) \qquad \Delta\psi = -\mathrm{tr}\,\nabla^2\psi = -\sum_{i=1}^{n}(\nabla^2\psi)(e_i, e_i)$$

- It is an L^2-symmetric non-negative operator, because

$$\int_M \langle \Delta\psi, \phi\rangle = \int_M \langle \nabla\psi, \nabla\phi\rangle$$

for sections of compact support

- It is an *elliptic* second order differential operator and so it has a real discrete non-bounded spectrum

- When $\Sigma M = M \times \mathbb{C}$, Δ is the usual Laplacian

Elliptic Differential Operators

• Lemma 2 Let $L : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator taking sections of a vector bundle E on sections of another vector bundle F on a compact Riemannian manifold M.

- Then both $\ker L$ and $\operatorname{coker} L$ are finite-dimensional and the *index* of L, defined by

  $$\operatorname{ind} L = \dim \ker L - \dim \operatorname{coker} L = \dim \ker L - \dim \ker L^*,$$

  where $L^* : \Gamma(F) \to \Gamma(E)$ is the formal adjoint of L with respect to the L^2-products, depends only on the homotopy class of L.

- If $E = F$ and the operator L is L^2-symmetric, then its spectrum is a sequence of real numbers and its eigenspaces are finite-dimensional and consist of smooth sections.

Dirac Bundles

- A simple geometrical tool

$$\gamma \in \Gamma(T^*M \otimes \mathrm{End}_{\mathbb{C}}(\Sigma M)) \qquad \gamma : TM \to \mathrm{End}_{\mathbb{C}}(\Sigma M)$$

allowing the definition of a first order operator

$$D = \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}$$

- Compatibility with Hermitian product and connections

$$\langle \gamma(Y)\psi, \eta\rangle = -\langle \psi, \gamma(Y)\eta\rangle \quad \nabla_X \gamma(Y)\psi = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X \psi$$

implies that D is L^2-symmetric

$$\int_M \langle D\psi, \phi\rangle = \int_M \langle \psi, D\phi\rangle, \qquad \forall \psi, \eta \in \Gamma_0(\Sigma M)$$

Dirac Bundles

• Compatibility between D and the rough Laplacian Δ comes from the Clifford relations

$$\gamma(X)\gamma(Y)+\gamma(Y)\gamma(X) = -2\langle X, Y\rangle \qquad X, Y \in \Gamma(TM)$$

and implies

$$D^2 = \Delta + \frac{1}{2}\sum_{i,j=1}^{n} \gamma(e_i)\gamma(e_j)R^{\Sigma M}(e_i, e_j)$$

• Consequence: Both D^2 and D are elliptic

• Consequence: If the manifold M is compact, then D has a real discrete spectrum tending to $+\infty$ and to $-\infty$

• $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ *is a* **Dirac bundle** *over* M, γ *is the Clifford multiplication and* D *is the Dirac operator*

Dirac Bundles

• Lemma 3 Let ΣM be a complex vector bundle over a Riemannian manifold M endowed with a Clifford multiplication $\gamma : TM \to \mathrm{End}_{\mathbb{C}}(\Sigma M)$. Then, there are a Hermitian metric $\langle\ ,\ \rangle$ and a unitary connection ∇ such that $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ is a Dirac bundle. Moreover, if we make the following changes

$$\langle\ ,\ \rangle \hookrightarrow \langle\ ,\ \rangle' = f^2 \langle\ ,\ \rangle, \qquad \nabla \hookrightarrow \nabla' = \nabla + d \log f + i\alpha,$$

where f is a positive smooth function on M and α is a real 1-form, then $(\Sigma M, \langle\ ,\ \rangle', \nabla', \gamma)$ is another Dirac bundle over the manifold M.

Dirac Bundles: New from Old

• Take a Dirac bundle $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ and a complex vector bundle $(E, \langle\ ,\ \rangle^E, \nabla^E)$ equipped with a Hermitian metric and a unitary metric connection and put

$$\Sigma' M = \Sigma M \otimes E \qquad \langle\ ,\ \rangle' = \langle\ ,\ \rangle \otimes \langle\ ,\ \rangle^E \qquad \nabla' = \nabla \otimes \nabla^E$$

• Define a new Clifford multiplication by

$$\gamma'(X)(\psi \otimes e) = (\gamma(X)\psi) \otimes e \qquad \psi \in \Gamma(\Sigma M), e \in \Gamma(E)$$

• Check that $(\Sigma' M, \langle\ ,\ \rangle', \nabla', \gamma')$ is another Dirac bundle called ΣM twisted by E

• If E is a complex line bundle, then twisting by E keeps the rank N unchanged

The Exterior Bundle as a Dirac Bundle

- Take as a complex vector bundle

$$\Sigma M = \Lambda^*_{\mathbb{C}}(M) = \bigoplus_{k=0}^{n} (\Lambda^k(M) \otimes \mathbb{C})$$

endowed with the Hermitian metric and the Levi-Cività connection induced from those of M

- Prove (use Lemma 1) that this definition

$$\gamma(X)\omega = X^\flat \wedge \omega - X \lrcorner\, \omega \qquad X \in \Gamma(TM), \omega \in \Gamma(\Lambda^*_{\mathbb{C}}(M))$$

provides a compatible Clifford multiplication

- Then $(\Lambda^*_{\mathbb{C}}(M), \langle\ ,\ \rangle, \nabla, \gamma)$ is a Dirac bundle with rank $N = 2^n$ and its Dirac operator satisfies

$$D = \sum_{i=1}^{n} e_i^\flat \wedge \nabla_{e_i} - \sum_{i=1}^{n} e_i \lrcorner\, \nabla_{e_i} = d + \delta \qquad D^2 = \Delta_H$$

The Exterior Bundle as a Dirac Bundle

- Hodge-de Rham Theorem If M is compact

$$\ker D = \ker \Delta_H \cong H^*(M,\mathbb{R}) = \bigoplus_{k=1}^{n} H^k(M,\mathbb{R})$$

- The curvature R^* of this Dirac operator is easy to compute for 1-forms and so

$$\int_M |(d+\delta)\omega|^2 = \int_M |\nabla\omega|^2 + \int_M \mathrm{Ric}\,(\omega,\omega) \qquad \omega \in \Gamma_0(\Lambda^1_{\mathbb{C}}(M))$$

- [Bochner Theorem] If M is a compact Riemannian manifold with positive Ricci curvature, then there are no non-trivial harmonic 1-forms on M. As a consequence the first Betti number of M vanishes

The Exterior Bundle as a Dirac Bundle

- If $\omega = df$ for a smooth function f, then

$$\int_M |\Delta f|^2 = \int_M |\nabla^2 f|^2 + \int_M \mathrm{Ric}\,(\nabla f, \nabla f)$$

- [Lichnerowicz-Obata Theorem] Let M be a compact Riemannian manifold of dimension n whose Ricci curvature satisfies $\mathrm{Ric} \geq \mathrm{Ric}_{\mathbb{S}^n(1)} = n - 1$ Then, the non-zero eigenvalues λ of the Laplacian operator of M acting on functions satify $\lambda \geq n$ The equality is attained if and only if M is isometric to an n-dimensional unit sphere

- For the equality, solve the Obata equation

$$\nabla^2 f = -f \langle \, , \, \rangle$$

The Exterior Bundle as a Dirac Bundle

- The Dirac-Euler operator $D = d + \delta$ does not preserve the degree of forms, but it does preserve the parity of the degree

- Consider the restrictions

$$D^{\text{even}} = D_{|\Gamma(\Lambda^{\text{even}}(M))} \qquad D^{\text{odd}} = D_{|\Gamma(\Lambda^{\text{odd}}(M))}$$

- They are elliptic operators and adjoint each other

- A First Index Theorem

$$\text{ind}\, D^{\text{even}} = \dim\ker D^{\text{even}} - \dim\ker D^{\text{odd}} = \sum_{k \text{ even}} b_k(M) - \sum_{k \text{ odd}} b_k(M) = \chi(M)$$

$$\text{ind}\, D^{\text{even}} = \int_M e(M)$$

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Antiholomorphic Exterior Bundle as a Dirac Bundle

- Suppose that M is Kähler with dimension $n = 2m$ and take

$$\Sigma M = \Lambda^{0,*}(M) = \bigoplus_{k=0}^{m} \Lambda^{0,k}(M) \subset \Lambda^{*}_{\mathbb{C}}(M)$$

endowed with the Hermitian metric and the Levi-Cività connection induced from those of M

- Modify the definition of γ in this way

$$\gamma^*(X)\omega = \sqrt{2}((X^\flat \wedge \omega)^{0,r+1} - X \lrcorner\, \omega) \qquad X \in \Gamma(TM), \omega \in \Gamma(\Lambda^{0,r}(M))$$

- Then $(\Lambda^{0,*}(M), \langle\ ,\ \rangle, \nabla, \gamma^*)$ is a Dirac bundle with rank $N = 2^m = 2^{\frac{n}{2}}$ and its Dirac operator satisfies

$$D^* = \sqrt{2}\left(\sum_{i=1}^{n} (e_i^\flat \wedge \nabla_{e_i})^{0,*} - \sum_{i=1}^{n} e_i \lrcorner\, \nabla_{e_i}\right) = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$$

Antiholomorphic Exterior Bundle as a Dirac Bundle

• Hodge-Dolbeault Theorem If M is a compact Kähler manifold

$$\ker D^* = \ker \Delta_{H|\Gamma(\Lambda^{0,*}(M))} \cong H^*(M, \mathcal{O}) = \bigoplus_{k=1}^{m} H^k(M, \mathcal{O})$$

where $\mathcal{O}$ is the sheaf of the holomorphic functions on M

• The curvature $R^{0,*}$ of this Dirac bundle is easy to compute on each degree and only depends on the Ricci curvature of the manifold M

• [Kodaira Theorem] If M is a compact Kähler manifold with dimension $n = 2m$ and positive Ricci curvature, then there are no non-trivial harmonic antiholomorphic q-forms on M with $q > 0$. As a consequence $H^q(M, \mathcal{O}) = 0$ for $0 < q \leq m$

Antiholomorphic Exterior Bundle as a Dirac Bundle

- The Dirac-Kähler operator $D^* = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ does not preserve the degree of forms, but it does preserve the parity of the degree

- Consider the restrictions

$$D^{*\text{even}} = D^*{}_{|\Gamma(\Lambda^{0,\text{even}}(M))} \qquad D^{*\text{odd}} = D^*{}_{|\Gamma(\Lambda^{0,\text{odd}}(M))}$$

- They are elliptic operators and adjoint each other

- A Second Index Theorem

$$\text{ind}\, D^{*\text{even}} = \sum_{q\ \text{even}} \dim H^q(M,\mathcal{O}) - \sum_{q\ \text{odd}} \dim H^q(M,\mathcal{O}) = \chi_{\mathcal{O}}(M)$$

the Todd genus of M

Rank and Dimension

• Exterior bundle: $N = 2^n$. Antiholomorphic exterior bundle: $N = 2^{\frac{n}{2}}$. Is there some general relation between the rank N of a Dirac bundle $(M, \langle\ ,\ \rangle, \nabla, \gamma)$ and the dimension n of the manifold M? Must come from $\gamma : TM \to \mathrm{End}_{\mathbb{C}}(\Sigma M)$

• Take $p \in M$. The *Clifford algebra* $\mathbb{C}\ell(T_pM)$ is the complex algebra spanned by the vectors of T_pM subjected to these definition relations

$$u \cdot v + v \cdot u = -2\langle u, v\rangle, \qquad u, v \in T_pM$$

It has complex dimension 2^n

• The Clifford relations satisfied by the Clifford multiplication γ mean exactly that it extends to a complex algebra homomorphism

$$\gamma_p : \mathbb{C}\ell(T_pM) \to \mathrm{End}_{\mathbb{C}}(\Sigma_p M), \quad \gamma_p(\lambda u_1 \cdots u_k) = \lambda \gamma_p(u_1) \cdots \gamma_p(u_k)$$

where $\lambda \in \mathbb{C}$ and $u_1, \ldots, u_k \in T_pM$

Rank and Dimension

- Order an orthomormal basis $\{e_1, \ldots, e_n\}$ of T_pM in this way

$$e_1, \ldots, e_k, e_{1^*}, \ldots, e_{k^*} \qquad k = \left[\frac{n}{2}\right]$$

and put

$$P_\alpha = \frac{1}{\sqrt{2}}\gamma_p(e_\alpha) - \frac{i}{\sqrt{2}}\gamma_p(e_{\alpha^*}), \quad Q_\alpha = \frac{1}{\sqrt{2}}\gamma_p(e_\alpha) + \frac{i}{\sqrt{2}}\gamma_p(e_{\alpha^*})$$

- The Clifford relations satisfied by γ_p give

$$P_\alpha P_\beta + P_\beta P_\alpha = Q_\alpha Q_\beta + Q_\beta Q_\alpha = 0, \quad P_\alpha Q_\beta + Q_\beta P_\alpha = -\delta_{\alpha\beta}$$

- See that $P = P_1 \cdots P_k \neq 0$ and choose $\psi = P\psi_0 \neq 0$. Then

$\psi, Q_{\alpha_1}\psi, (Q_{\alpha_1}Q_{\alpha_2})\psi, \ldots, (Q_{\alpha_1}Q_{\alpha_2}\cdots Q_{\alpha_{k-1}})\psi, (Q_1 Q_2 \cdots Q_k)\psi \in \Sigma_p M$
with $1 \leq \alpha_1 < \ldots < \alpha_l \leq k$, are linearly independent.

Rank and Dimension

• Proposition Let ΣM be a rank N Dirac bundle on an n-dimensional Riemannian manifold M. Then we have the inequality

$$N \geq 2^{[\frac{n}{2}]},$$

and the equality is attained if and only if the Clifford multiplication

$$\gamma_p : \mathbb{C}\ell(T_pM) \to \mathrm{End}_{\mathbb{C}}(\Sigma_p M)$$

at each point p of the manifold provides a complex algebra epimorphism. In fact, in this case, γ_p is an isomorphism when n is even and, when n is odd, γ_p is an isomorphism when it is restricted to the Clifford algebra of any hyperplane of the tangent space T_pM

Minimal Rank: Spinor Bundles and Spin^c Manifolds

- *A Dirac bundle* ΣM *with minimal rank* $N = 2^{[\frac{n}{2}]}$ *is called a* **spinor bundle** *and its sections* $\psi \in \Gamma(\Sigma M)$ *are called spinor fields*

- *A Riemannian manifold* M *which supports a spinor bundle over it will be said to be a* **spin^c manifold**

- *A* **spin^c structure** *on a spin^c manifold* M *is an isomorphism class of spinor bundles* ΣM

- The complex exterior bundle $\Lambda^*_{\mathbb{C}}(M)$ over a Riemannian manifold M **is not** a spinor bundle ($N = 2^n > 2^{[\frac{n}{2}]}$)

- The antiholomorphic exterior bundle $\Lambda^*_{\mathbb{C}}(M)$ over a Kähler manifold M **is** a spinor bundle ($N = 2^m = 2^{\frac{n}{2}} = 2^{[\frac{n}{2}]}$). Each Kähler manifold is a spin^c manifold

Minimal Rank: Spinor Bundles and Spin^c Manifolds

• Proposition Let $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ be a spinor bundle over a spin^c manifold M

- Metric and connection uniqueness

$$\langle\ ,\ \rangle' = f^2\langle\ ,\ \rangle, \qquad \nabla' = \nabla + d\log f + i\alpha$$

  for a positive smooth function f and a real 1-form α

- Dirac bundles uniqueness

$$\Sigma' M \cong \Sigma M \otimes E$$

  If $\Sigma' M$ is another spinor bundle, then E is a line bundle

- $\mathcal{A}_{\Sigma M} = \mathrm{Hom}_{\gamma}(\Sigma M, \overline{\Sigma M})$ has rank one. It is called the *auxiliary line bundle* of ΣM. If L is an arbitrary line bundle, we have

$$\mathcal{A}_{\Sigma M \otimes L} = \mathcal{A}_{\Sigma M} \otimes L^{-2}$$

Minimal Rank: Spinor Bundles and Spin^c Manifolds

- Example: $\mathcal{A}_{\Lambda^{0,*}(M)} \cong K_M^{-1} = \Lambda^{0,m}(M)$

- Consequence: Spin^c structures on a spin^c manifold are parametrized by $H^2(M,\mathbb{Z})$

- Proposition Clifford multiplication uniqueness

$$\gamma' = P\gamma P^{-1} \ (n \text{ even}) \text{ or } \gamma' = \pm P\gamma P^{-1} \ (n \text{ odd})$$

for an isometry field $P \in \Gamma(\mathrm{Aut}_{\mathbb{C}}(\Sigma M))$

- All these uniqueness results follow from the fundamental fact characterizing spinor bundles among all Dirac bundles: any complex endomorphism of any of its fibers commuting with all the Pauli matrices must be a complex multiple of the identity

Spinor Bundles with Trivial Auxiliary Bundle

• If $\mathcal{A}_{\Sigma M}$ is trivial, any $\phi \in \Gamma(\mathcal{A}_{\Sigma M})$ nowhere vanishing section satisfies $\phi^2 = hI$ for a real function h

• Proposition Let ΣM be a spinor bundle over a spin^c manifold M. Then the auxiliary line bundle $\mathcal{A}_{\Sigma M}$ is trivial if and only if there is an isometric **real** o **quaternionic** structure on the spinor bundle **commuting** with γ, that is, a complex antilinear bundle isometry $\theta : \Sigma M \to \Sigma M$ with $\theta^2 = \pm I$. This structure is **parallel for exactly one** of the compatible connections

• On a given spin^c manifold M there is a spinor bundle ΣM with trivial auxiliary line bundle iff there is a spinor bundle whose auxiliary line bundle has a squared root

Trivial Auxiliary Bundle: Spin Manifolds

- *A spin^c manifold M which admits a spinor bundle with trivial auxiliary line bundle will be said to be a* **spin manifold**

- *A* **spin structure** *on a spin manifold M is an isomorphism class of pairs $(\Sigma M, \theta)$ consisting of a spinor bundle ΣM and a quaternionic or real structure θ defined on it. The only connection on ΣM parallelizing θ is called the* **spin Levi-Cività** *connection*

- The antiholomorphic exterior bundle $\Lambda^{0,*}(M)$ on a Kähler manifold M determines a spin structure iff its canonical line bundle K_M is trivial, that is, M is Calabi-Yau. A Kähler manifold is a spin manifold iff K_M is a square iff $[c_1(M)]_{\text{mod } 2} = 0$

- Spin structures on a spin manifold are parametrized by $H^1(M, \mathbb{Z}_2)$

Topological Consequences

- If M is a spin^c manifold, the cohomology class

$$[c_1(\mathcal{A}_{\Sigma M})]_{\mathsf{mod}\,2} \in H^2(M, \mathbb{Z}_2)$$

does not depend on the chosen spinor bundle ΣM

- The obstructions for the auxiliary line bundle to have a squared root and for the bundle of oriented orthonormal frames to have a twofold covering principal bundle with structure group $Spin(n)$ coincide, that is,

$$[c_1(\mathcal{A}_{\Sigma M})]_{\mathsf{mod}\,2} = w_2(M)$$

Topological Consequences

- **Theorem** Let M be an oriented Riemannian manifold. If M admits a spinor bundle, then its second Stiefel-Whitney cohomology class satisfies

$$w_2(M) \in \text{im}\left(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z}_2)\right)$$

If M admits a spinor bundle with trivial auxiliary line bundle, that is, if it admits a spinor bundle with a real or quaternionic structure, that is, if it admits a spin structure, then

$$w_2(M) = 0$$

- These necessary topological conditions are also sufficient

Geometrical Consequences

• On any Dirac bundle, compatibility between Clifford multiplication and connections

$$\nabla_Y \gamma(Z)\psi = \gamma(\nabla_Y Z)\psi + \gamma(Z)\nabla_Y \psi \qquad Y, Z \in \Gamma(TM), \psi \in \Gamma(\Sigma M)$$

implies (taking derivatives and skew-symmetrizing) compatibility between Clifford multiplication and curvatures

$$R^{\Sigma M}(X,Y)(\gamma(Z)\psi) = \gamma(R(X,Y)Z)\psi + \gamma(Z)(R^{\Sigma M}(X,Y)\psi)$$

where $X, Y, Z \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$

• You can check that this another operator

$$R^0(X,Y)\psi = \frac{1}{4}\sum_{i,j=1}^{n} R(X,Y,e_i,e_j)\gamma(e_i)\gamma(e_j)\psi$$

satisfies the same compatibility as $R^{\Sigma M}$ does

Geometrical Consequences

• Then, the difference $R' = R^{\Sigma M} - R^0$ satisfies

$$R'(X,Y)\gamma(Z)\psi = \gamma(Z)R'(X,Y)\psi \qquad X,Y,Z \in \Gamma(TM), \psi \in \Gamma(\Sigma M)$$

and so commutes with the Pauli matrices

• Consequence: If ΣM is a spinor bundle, this difference R' must be a scalar

$$R^{\Sigma M}(X,Y) = \frac{1}{4}\sum_{i,j=1}^{n} R(X,Y,e_i,e_j)\gamma(e_i)\gamma(e_j) + i\alpha(X,Y)$$

for $X, Y \in \Gamma(TM)$ and where α is a real two-form on M

• If $\theta \in \Gamma(\mathcal{A}_{\Sigma M})$, since we have $R^{\mathcal{A}_{\Sigma M}}(X,Y)\theta = [R^{\Sigma M}(X,Y), \theta]$ and θ commutes with the first addend,

$$R^{\mathcal{A}_{\Sigma M}}(X,Y) = 2i\alpha(X,Y) \qquad X,Y \in \Gamma(TM)$$

Geometrical Consequences

• Proposition The curvature operator of a spinor bundle ΣM over a Riemannian manifold M

$$R^{\Sigma M}(X,Y) = \frac{1}{4}\sum_{i,j=1}^{n} R(X,Y,e_i,e_j)\gamma(e_i)\gamma(e_j) + \frac{1}{2}R^{\mathcal{A}_{\Sigma M}}(X,Y)$$

is completely determined by the Riemannian curvature of M and the curvature (imaginary valued) two-form of the auxiliary line bundle $\mathcal{A}_{\Sigma M}$

• [Schrödinger-Lichnerowicz formula] If D is the Dirac operator of a spinor bundle, then

$$D^2 = \Delta + \frac{1}{4}S + \frac{1}{4}\sum_{i,j=1}^{n} R^{\mathcal{A}_{\Sigma M}}(e_i,e_j)\gamma(e_i)\gamma(e_j)$$

where S is the scalar curvature of M and $R^{\mathcal{A}_{\Sigma M}}$ the curvature of the auxiliary line bundle

Geometrical Consequences

• Take $\psi \in \Gamma(\Sigma M)$ in the spinor bundle of a spin structure ΣM on a compact spin manifold M. Then

$$\int_M |D\psi|^2 = \int_M |\nabla\psi|^2 + \frac{1}{4}\int_M S|\psi|^2$$

• [Lichnerowicz Theorem] If M is a compact spin manifold with positive scalar curvature, then there are no non-trivial harmonic spinor fields on any spin structure of M. If we weaken the curvature assumption into non-negative scalar curvature, we have that all harmonic spinor fields must be parallel.

• [Wang Classification] The only simply-connected spin manifolds carrying non-trivial parallel spinor fields are Calabi-Yau (including hyper-Kähler and flat) manifolds, associative seven-dimensional manifolds and Cayley eight-dimensional manifolds (all of them Ricci-flat)

An Introduction to the Dirac Operator in Riemannian Geometry

S. Montiel

Universidad de Granada

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The Index Theorem

• [Hitchin Ph.D. Thesis] The kernel of the Dirac operator of a spinor bundle ΣM on a compact spin^c manifold has no topological meaning and its index is zero, but...

• The complex volume element $\omega = i^{\left[\frac{n+1}{2}\right]}\gamma(e_1)\cdots\gamma(e_n)$ is a parallel section of $\mathrm{End}_{\mathbb{C}}(\Sigma M)$ with $\omega^2 = I$ which, **when** n **is even**, anti-commutes with the Clifford multiplication and so decomposes the spinor bundle and the Dirac operator

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M \qquad D = D^+ \oplus D^-$$

into *chiral* ± 1-eigenbundles with the same rank and the corresponding restrictions which are adjoint each other

• $\dim\ker D = \dim\ker D^+ + \dim\ker D^-$ has no topological meaning, but $\mathrm{ind}\, D^+ = \dim\ker D^+ - \dim\ker D^-$...

The Index Theorem

- [Atiyah-Singer Index Theorem]

$$\text{ind } D_E^+ = \int_M \text{ch}(E) e^{\frac{1}{2}c_1(\mathcal{A}_{\Sigma M})} \widehat{\mathbb{A}}(M)$$

for the Dirac operator of a spinor bundle ΣM twisted by a complex vector bundle E, where

$$\text{ch}(E) = \text{ch}(c_1(E), \ldots, c_l(E)) \qquad \widehat{\mathbb{A}}(M) = \hat{\mathcal{A}}(p_1(M), \ldots, p_k(M))$$

$$\text{ch}(c_1, \ldots, c_l) = \sum_{i=1}^{l} e^{x_i} \qquad \hat{\mathcal{A}}(p_1, \ldots, p_k) = \prod_{j=1}^{k} \frac{y_j}{2 \sinh \frac{y_j}{2}}$$

σ_k stands for the k-th symmetric elementary polynomial, and

$$\sigma_i(x_1, \ldots, x_l) = c_i, \qquad \sigma_j(y_1^2, \ldots, y_k^2) = p_j$$

where $c_i(E) \in H^{2i}(M, \mathbb{Z})$ are the Chern classes of E, $p_j(M) \in H^{4j}(M, \mathbb{Z})$ the Pontrjagin classes of M, and $c_1(\mathcal{A}_{\Sigma M}) \in H^2(M, \mathbb{Z})$ the Chern class of the auxiliary line bundle

The Index Theorem: Aplications

• If $n = \dim M = 2$, then

$$\mathrm{ch}(E) = l + c_1(E), \qquad e^{\frac{1}{2}c_1(\mathcal{A}_{\Sigma M})} = 1 + \frac{1}{2}c_1(\mathcal{A}_{\Sigma M}), \qquad \widehat{\mathbb{A}}(M) = 1$$

• [Index Theorem for Surfaces]

$$\mathrm{ind}\, D_E^+ = \int_M \left(\frac{l}{2} c_1(\mathcal{A}_{\Sigma M}) + c_1(E) \right)$$

• Spin case (trivial $\mathcal{A}_{\Sigma M}$ and E): $\mathrm{ind}\, D_E^+ = 0$

• The case of the antiholomorphic exterior bundle ($\Sigma M = \Lambda^{0,*}(M)$ and $\mathcal{A}_{\Sigma M} = K_M^{-1}$)

$$\dim H^0(M, \mathcal{O}(E)) - \dim H^1(M, \mathcal{O}(E)) = l(1 - g(M)) + \int_M c_1(E)$$

which is the Riemann-Roch theorem

The Index Theorem: Aplications

- If $n = \dim M = 4$ and $\mathcal{A}_{\Sigma M}$ is trivial, then

$$\mathrm{ch}(E) = l + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) \qquad \widehat{\mathbb{A}}(M) = 1 - \frac{1}{24}p_1(M)$$

- [Index Theorem for Spin Four-Manifolds]

$$\mathrm{ind}\, D_E^+ = \int_M \left(\frac{1}{2}c_1(E)^2 - c_2(E) - \frac{l}{24}p_1(M)\right)$$

- Case $E = \Sigma M$ ($l = 4$, $c_1(E) = 0$ because of the existence of the structure θ and $p_1(M) = -2c_2(E)$ because of $TM \otimes \mathbb{C} \cong \Sigma^+ M \otimes \Sigma^- M$)

- Therefore

$$\mathrm{ind}\, D_{\Sigma M}^+ = \frac{1}{3}\int_M p_1(M)$$

The Index Theorem: Aplications

• If $n = \dim M = 4$ and $\mathcal{A}_{\Sigma M}$ is trivial, we had

$$\text{ind}\, D^{+}_{\Sigma M} = \frac{1}{3}\int_M p_1(M)$$

• But $\Sigma M \otimes E = \Sigma M \otimes \Sigma M \cong \text{End}_{\mathbb{C}}(\Sigma M) \overset{\gamma}{\cong} \Lambda^*_{\mathbb{C}}(M)$ and so $D_E = d + \delta$

• See that $\omega \in \text{End}_{\mathbb{C}}(\Lambda^*_{\mathbb{C}}(M))$ coincides with the Hodge $*$ up to a sign

• [Hirzebruch Signature Theorem] Let M be a compact (spin) manifold of dimension four. Then, the signature of M is related with its first Pontrjagin class as follows

$$\sigma(M) = b_2^{+}(M) - b_2^{-}(M) = \frac{1}{3}\int_M p_1(M)$$

The Index Theorem: Aplications

- [Index Theorem for Spin Four-Manifolds II]

$$\text{ind } D_E^+ = -\frac{l}{8}\sigma(M) + \int_M \left(\frac{1}{2}c_1(E)^2 - c_2(E)\right)$$

- Take E as a trivial line bundle, then

$$\text{ind } D^+ = -\frac{1}{8}\sigma(M)$$

and remember that the quaternionic structure preserves $\Sigma^+ M$ and commutes with the Dirac operator

- [Rochlin Theorem] Let M be a four-dimensional compact spin manifold. Then, the signature of M is divisible by 16

- There are simply-connected compact topological four-manifolds with $w_2(M) = 0$ and $\sigma(M) = 8$ (!?)

The Index Theorem: Aplications

• [Integrality of the Â-genus] We define the Â-genus of an even-dimensional compact manifold as the rational number

$$\hat{A}(M) = \int_M \hat{\mathbb{A}}(M)$$

Then, if M is spin its Â-genus is an integer number

• [Lichnerowicz Theorem] Let M be a compact spin Riemannian manifold with positive scalar curvature. Then the Â-genus of M vanishes

• Notice that $\mathbb{C}P^2$ is a compact non-spin four-manifold with

$$\hat{A}(\mathbb{C}P^2) = -\frac{1}{8}\sigma(\mathbb{C}P^2) = -\frac{1}{8}$$

and admits a metric with positive scalar curvature

The Spectrum of the Dirac Operator

- Bochner/Lichnerowicz-Obata=Lichnerowicz/Friedrich-Bär
- If M is a compact spin manifold, the integral Schrödinger-Lichnerowicz formula gave

$$\int_M \left(|D\psi|^2 - |\nabla\psi|^2 - \frac{1}{4} S|\psi|^2 \right) = 0$$

for each $\psi \in \Gamma(\Sigma M)$

- Use this Schwarz inequality

$$|D\psi|^2 = \left| \sum_{i=1}^{n} \gamma(e_i)\nabla_{e_i}\psi \right|^2 \leq \left(\sum_{i=1}^{n} |\gamma(e_i)\nabla_{e_i}\psi| \right)^2 = \left(\sum_{i=1}^{n} |\nabla_{e_i}\psi| \right)^2 \leq n|\nabla\psi|^2$$

and get the Friedrich inequality

$$\int_M \left(|D\psi|^2 - \frac{n}{4(n-1)} S|\psi|^2 \right) \geq 0$$

The Spectrum of the Dirac Operator

• [Friedrich Theorem] Let M be a compact spin manifold of dimension n whose scalar curvature satisfies $S \geq S_{\mathbb{S}^n(1)} = n(n-1)$ Then, the eigenvalues λ of the Dirac operator of any spin structure of M satify $|\lambda| \geq \frac{n}{2}$

• For the equality, solve the Killing spinor equation

$$\nabla\psi = \mp\gamma\psi$$

• [Bär Classification] Let M be a compact simply connected spin Riemannian manifold admitting a non-trivial Killing spinor field. Then M is isometric to a **sphere** $\mathbb{S}^n$, or to an **Einstein-Sasakian** manifold, or to a **six-dimensional nearly-Kähler** non-Kähler manifold, or to a **seven-dimensional associative** manifold

The Spectrum of the Dirac Operator

• Conformal change of metric: If $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ is a spinor bundle on $(M, \langle\ ,\ \rangle)$, then $(\Sigma M, \langle\ ,\ \rangle, \nabla^*, \gamma^*)$ with

$$\nabla^* = \nabla - \frac{1}{2}\gamma(\cdot)\gamma(\nabla u) - \frac{1}{2}\langle \cdot, \nabla u\rangle \quad \gamma^* = e^u \gamma$$

is a spinor bundle on $(M, \langle\ ,\ \rangle^*)$ with

$$\langle\ ,\ \rangle^* = e^{2u}\langle\ ,\ \rangle$$

• Conformal covariance of the Dirac operator:

$$D^*(e^{-\frac{n-1}{2}u}\psi) = e^{-\frac{n+1}{2}u} D\psi, \qquad \psi \in \Gamma(\Sigma M)$$

• Recipe: Write the Friedrich inequality for the conformal metric $\langle\ ,\ \rangle^*$ and use this conformal covariance

The Spectrum of the Dirac Operator

- Result: A new conformal Friedrich inequality

$$\int_M e^{-u}\left\{|D\psi|^2 - \frac{n}{4(n-1)}S^* e^{2u}|\psi|^2\right\} \geq 0$$

valid for $\psi \in \Gamma(\Sigma M)$ and **for all smooth function** $u \in C^\infty(M)$

- Choose u to do $S^* e^{2u}$ constant taking into account that

$$S^* e^{2u} = \begin{cases} e^{-\frac{n-2}{2}u}\mathcal{Y}e^{\frac{n-2}{2}u} \text{ if } n \geq 3 \\ -2\Delta u + 2K \text{ if } n = 2 \end{cases}$$

to become constant, where $\mathcal{Y} = \frac{4(n-1)}{n-2}\Delta + S$ is the Yamabe operator

- A suitable choice gives

$$S^* e^{2u} = \begin{cases} \mu_1(\mathcal{Y}) \text{ (the first eigenvalue of } \mathcal{Y}) \text{ if } n \geq 3 \\ \frac{4\pi\chi(M)}{A(M)} \text{ if } n = 2 \end{cases}$$

The Spectrum of the Dirac Operator

• [Hijazi Theorem] Suppose that λ is an eigenvalue of the Dirac operator of a compact spin Riemannian manifold M with dimension $n \geq 3$ which admits a conformal metric with positive scalar curvature. Then

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1(\mathcal{Y})$$

where $\mu_1(\mathcal{Y})$ is the first eigenvalue of the Yamabe operator of M

• [Bär Theorem] If M is a compact spin surface of genus zero and λ is an eigenvalue of its Dirac operator, then

$$\lambda^2 \geq \frac{4\pi}{A(M)}$$

where $A(M)$ is the area of M

• If M is a compact spin manifold **with non-empty boundary** $\partial M = \mathcal{S}$, the integral Schrödinger-Lichnerowicz formula has a boundary term

$$\int_M \left(|D\psi|^2 - |\nabla\psi|^2 - \frac{1}{4} S |\psi|^2 \right) = - \int_{\mathcal{S}} \langle \gamma(N) D\psi + \nabla_N \psi, \psi \rangle$$

where $\psi \in \Gamma(\Sigma M)$ and N is the inner unit normal

• Restriction of the spinor bundle: If $(\Sigma M, \langle\ ,\ \rangle, \nabla, \gamma)$ is the spinor bundle on M, then $(\Sigma M_{|\mathcal{S}}, \langle\ ,\ \rangle, \nabla\!\!\!/, \gamma\!\!\!/)$ with

$$\nabla\!\!\!/ = \nabla - \frac{1}{2}\gamma(A\cdot)\gamma(N) \qquad \gamma\!\!\!/ = \gamma\gamma(N)$$

is a Dirac bundle on the hypersurface $\mathcal{S}$

• [Reilly Inequality] Rewrite the bundary term using the structure of the restricted Dirac bundle and use the same Schwarz inequality as in the boundary free case. Then

$$\int_M \left(\frac{1}{4}S|\psi|^2 - \frac{n}{n+1}|D\psi|^2\right) \leq \int_{\mathcal{S}} \left(\langle \not{D}\psi, \psi\rangle - \frac{n}{2}H|\psi|^2\right)$$

for $\psi \in \Gamma(\Sigma M)$ and where H is the (inner) mean curvature of the n-dimensional boundary

• [Reilly-Witten Trick] When the bulk manifold M has non-negative scalar curvature, solve a boundary problem

$$D\psi = 0 \text{ on } M \qquad \pi_+\psi_{|\mathcal{S}} = \pi_+\phi \text{ along } \mathcal{S}$$

for a given spinor field ϕ on the boundary and a suitable π_+ (usually needed $H \geq 0$)

The Spectrum of the Dirac Operator

• Witten proof of the positive mass: M is a complete non-compact asymptotically Euclidean three-manifold with non-negative scalar curvature. Take $\mathcal{S}_r$ as the sphere $|x| = r$ and choose ϕ_r as a unit Killing spinor for the round metric. Then

$$0 \leq \lim_{r \to +\infty} \int_{\mathcal{S}_r} \left(\langle \not{D} \psi_r, \psi_r \rangle - H_r |\psi_r|^2 \right) = 4\pi m$$

where m is a constant asociated to the end: its **mass**

• Other choice: M is a compact spin manifold with non-negative scalar curvature and the hypersurface boundary $\mathcal{S}$ has non-negative (inner) mean curvature. Take ϕ as an eigen-spinor for the eigenvalue of $\not{D}$ with the least absolute value

The Spectrum of the Dirac Operator

• [Hijazi-M-Zhang extrinsic comparison] Let $\mathcal{S}$ be a hypersurface bounding a domain in a spin manifold of dimension $n+1$with non-negative scalar curvature and suppose that the mean curvature of $\mathcal{S}$ satisfies $H \geq H_{\mathbb{S}^n(1)} = 1$ Then, the eigenvalues λ of the Dirac operator of the induced spin structure of $\mathcal{S}$ satify $|\lambda| \geq \frac{n}{2}$ and the equality holds iff the eigenspinors associated to $\pm\frac{n}{2}$ are restrictions to $\mathcal{S}$ of parallel spinors on the bulk manifold M