Geometry, Integrability and Quatization – June 8-13, 2007

Finding Lie Symmetries of PDEs with *MATHEMATICA*: Applications to Nonlinear Fiber Optics

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Plan of Presentation

1. *MATHEMATICA* package for finding Lie symmetries of PDE
    1.1. Block-scheme and algorithm
    1.2. Input and output
    1.3. Tracing the evaluation
    1.4. Trial run
2. Applications to nonlinear fiber optics
    2.1. Physical model
    2.2. Results obtained
3. Conclusion

System of PDE

$$F_k\left(x,u,u^{(1)},\dots,u^{(n)}\right)=0,\quad k=1,\ 2,\ \dots,\ l \qquad (\Delta)$$

Creating Defining System

MATHEMATICA

$$\mathrm{pr}^{(n)}X\left[\mathrm{F}\left(\mathrm{z}^{(\mathrm{n})}\right)\right]=0 \ \text{ for } \ z^{(n)}\in\Delta_F$$

Solving Defining System

$$\xi^i=\xi^i(x,u),\ \eta^\alpha=\eta^\alpha(x,u)$$

Basic Infinitesimal Generators

$$X_\nu=\sum_{i=1}^{p}\xi_\nu^i(x,u)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\eta_\nu^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

Solving the Lie Equation

$$\frac{df}{da}=\xi(f,\varphi),\quad f_{|a=0}=x$$

$$\frac{d\varphi}{da}=\eta(f,\varphi),\quad \varphi_{|a=0}=u$$

Symmetry Group of Δ

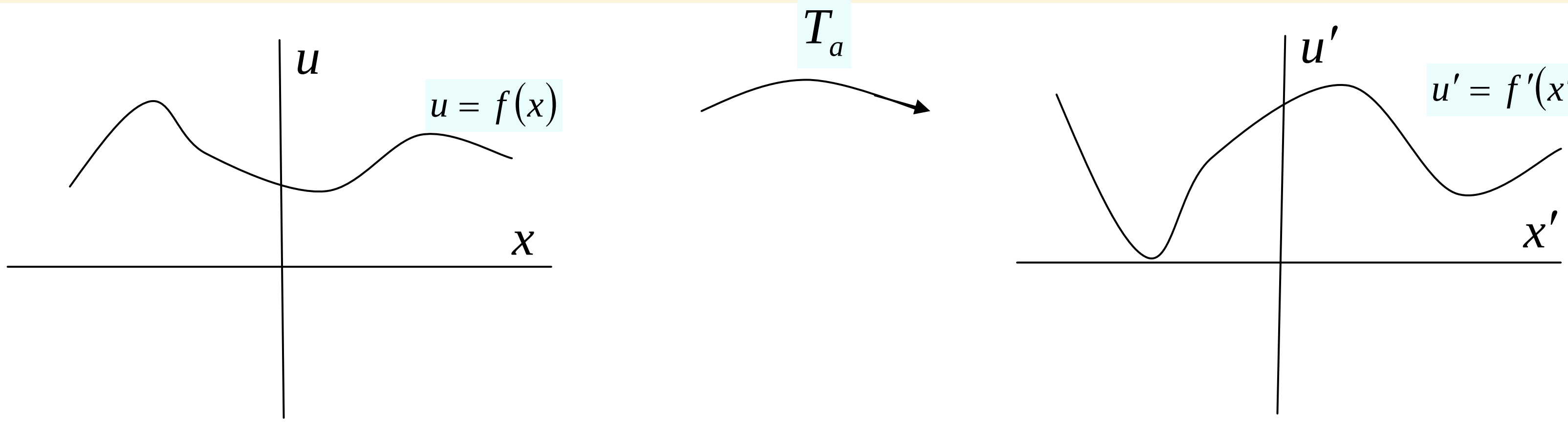
$$G^r=\left\{T_a \,\middle|\, a\in\Omega\subset R^r,0\in\Omega\right\}$$

Lie Group of Symmetry Transformations

$$F_k\left(x,u,u^{(1)},\ldots,u^{(n)}\right)=0,\quad k=1,\ 2,\ \ldots,\ l \qquad (\Delta)$$

$$G=\left\{T_a \mid a\in\delta\subset R, 0\in\delta\right\}$$

Each solution of Δ after transformation of the group G remains a solution of Δ .



If f is a solution of Δ then $f' = T_a \cdot f$ **is also a solution of Δ .**

The system of PDE and the Prolonged Space

$$F_k\left(x,u,u^{(1)},\dots,u^{(n)}\right)=0,\quad k=1,\ 2,\ \dots,\ l \qquad (\Delta)$$

$$x=\left(x^1,\dots,x^p\right)\in R^p \qquad u=\left(u^1,\dots,u^q\right)\in R^q$$

$$u^{(s)}=\left\{u^{\alpha}_{j_1,\dots,j_s}\equiv\frac{\partial u^{\alpha}}{\partial x_{j_1},\dots,\partial x_{j_s}}\middle|\alpha=1,\dots,q;\ j_k=1,\dots,p;k=1,\dots,s\right\}$$

$z\in Z=R^p\times R^q$ ⟹ $z^{(n)}=\left(x,u,u^{(1)},u^{(2)},\dots,u^{(n)}\right)\in Z^n$

$Z^{(n)}$ **is the** n^{th} **prolongation of the space** Z

$$\Delta_F=\left\{z^{(n)}\in Z^{(n)}\middle|\ F\left(z^{(n)}\right)=0\right\}\subset Z^{(n)}$$

The system Δ **is considered as a sub-manifold** Δ_F **in the prolonged space** $Z^{(n)}$.

n^{th} prolongation of the Infinitesimal Generator X

$$X=\sum_{i=1}^{p}\xi^{i}(x,u)\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q}\eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}$$

$$\mathrm{pr}^{(n)}X=X+\sum_{i=1}^{p}\sum_{\alpha=1}^{q}\varsigma_{i}^{\alpha}\frac{\partial}{\partial u_{i}^{\alpha}}+\cdots+\sum_{j_1=1}^{p}\cdots\sum_{j_n=1}^{p}\sum_{\alpha=1}^{q}\varsigma_{j_1\ldots j_n}^{\alpha}\frac{\partial}{\partial u_{j_1\cdots j_n}^{\alpha}}$$

$$\varsigma_{i}^{\alpha}=D_i\left(\eta^{\alpha}\right)-\sum_{s=1}^{p}u_s^{\alpha}D_i\left(\xi^{s}\right)$$

$$\varsigma_{j_1\ldots j_k}^{\alpha}=D_{j_k}\left(\varsigma_{j_1\ldots j_{k-1}}^{\alpha}\right)-\sum_{s=1}^{p}u_{j_1\ldots j_{k-1}s}^{\alpha}D_{j_k}\left(\xi^{s}\right)$$

$$D_i=\frac{\partial}{\partial x^{i}}+\sum_{\alpha=1}^{q}u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+\sum_{j=1}^{p}\sum_{\alpha=1}^{q}u_{ji}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\cdots+\sum_{j_1}^{p}\cdots\sum_{j_{n-1}=1}^{p}\sum_{\alpha=1}^{q}u_{j_1\cdots j_{n-1}i}^{\alpha}\frac{\partial}{\partial u_{j_1\cdots j_{n-1}}^{\alpha}}$$

The Infinitesimal Criterion and the Defining System

$$X = \sum_{i=1}^{p} \xi^{i}(x,u)\frac{\partial}{\partial x^{i}} + \sum_{\alpha=1}^{q} \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}$$

$$\mathrm{pr}^{(n)}X = X + \sum_{i=1}^{p}\sum_{\alpha=1}^{q} \varsigma_{i}^{\alpha}\frac{\partial}{\partial u_{i}^{\alpha}} + \cdots + \sum_{j_1=1}^{p}\cdots\sum_{j_n=1}^{p}\sum_{\alpha=1}^{q} \varsigma_{j_1\ldots j_n}^{\alpha}\frac{\partial}{\partial u_{j_1\cdots j_n}^{\alpha}}$$

G^{1} is a Lie group of symmetry transformations of the system of PDE Δ with the infinitesimal generator X.

⟷

The infinitesimal criterion holds.

$$\mathrm{pr}^{(n)}X\left[\mathrm{F}\left(\mathrm{z}^{(\mathrm{n})}\right)\right] = 0 \text{ for } z^{(n)} \in \Delta_{F}$$

Defining System

System of PDE

$$F_k\left(x,u,u^{(1)},\dots,u^{(n)}\right)=0,\quad k=1,\ 2,\ \dots,\ l \qquad (\Delta)$$

Creating Defining System

MATHEMATICA

$$\mathrm{pr}^{(n)}X\left[\mathrm{F}\left(\mathrm{z}^{(\mathrm{n})}\right)\right]=0 \text{ for } z^{(n)}\in\Delta_F$$

Solving Defining System

$$\xi^i=\xi^i(x,u),\ \eta^\alpha=\eta^\alpha(x,u)$$

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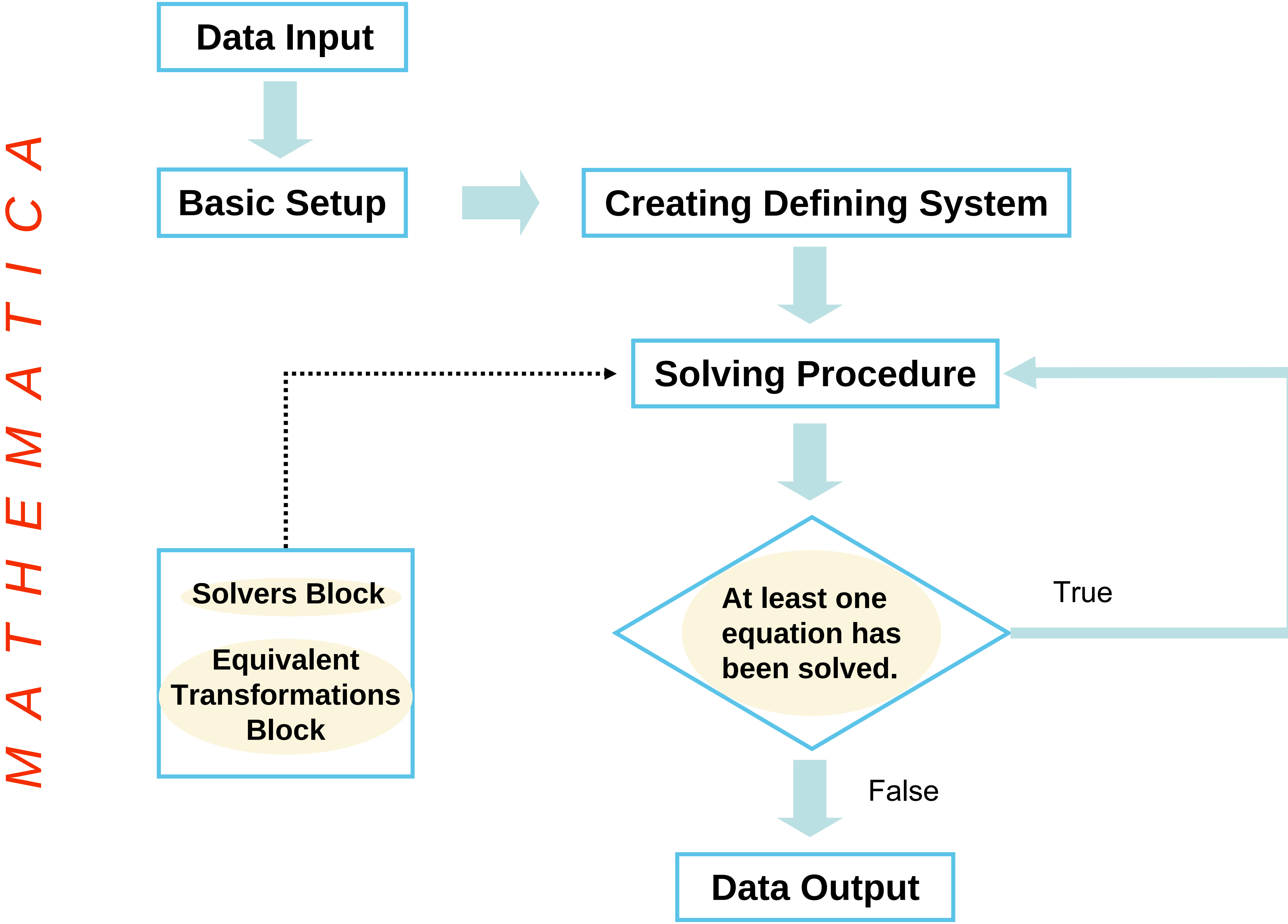
Symmetry Group of Δ

$$G^r=\left\{T_a \,\middle|\, a\in\Omega\subset R^r,0\in\Omega\right\}$$


MATHEMATICA
Data Input
Basic Setup
Creating Defining System
Solving Procedure
Solvers Block
Equivalent Transformations Block
At least one equation has been solved.
True
False
Data Output

Data Input

PDE $\equiv \{F_1 = 0, \ldots, F_l = 0\}$

indvar $\equiv \{x^1, \ldots, x^p\}$

depvar $\equiv \{u^1, \ldots, u^q\}$

deriv $\equiv \{u^{\alpha}_{j_1,\ldots,j_s}\}$

- ***Data Input is data about the considered PDE.***

Basic Set-Up

LHS $\equiv \{F_1, \ldots, F_l\}$

Man $\equiv \Delta_F$

InfGen $\equiv \{\xi^1(x,u), \ldots, \xi^p(x,u), \ldots, \eta^1(x,u), \ldots, \eta^q(x,u)\}$

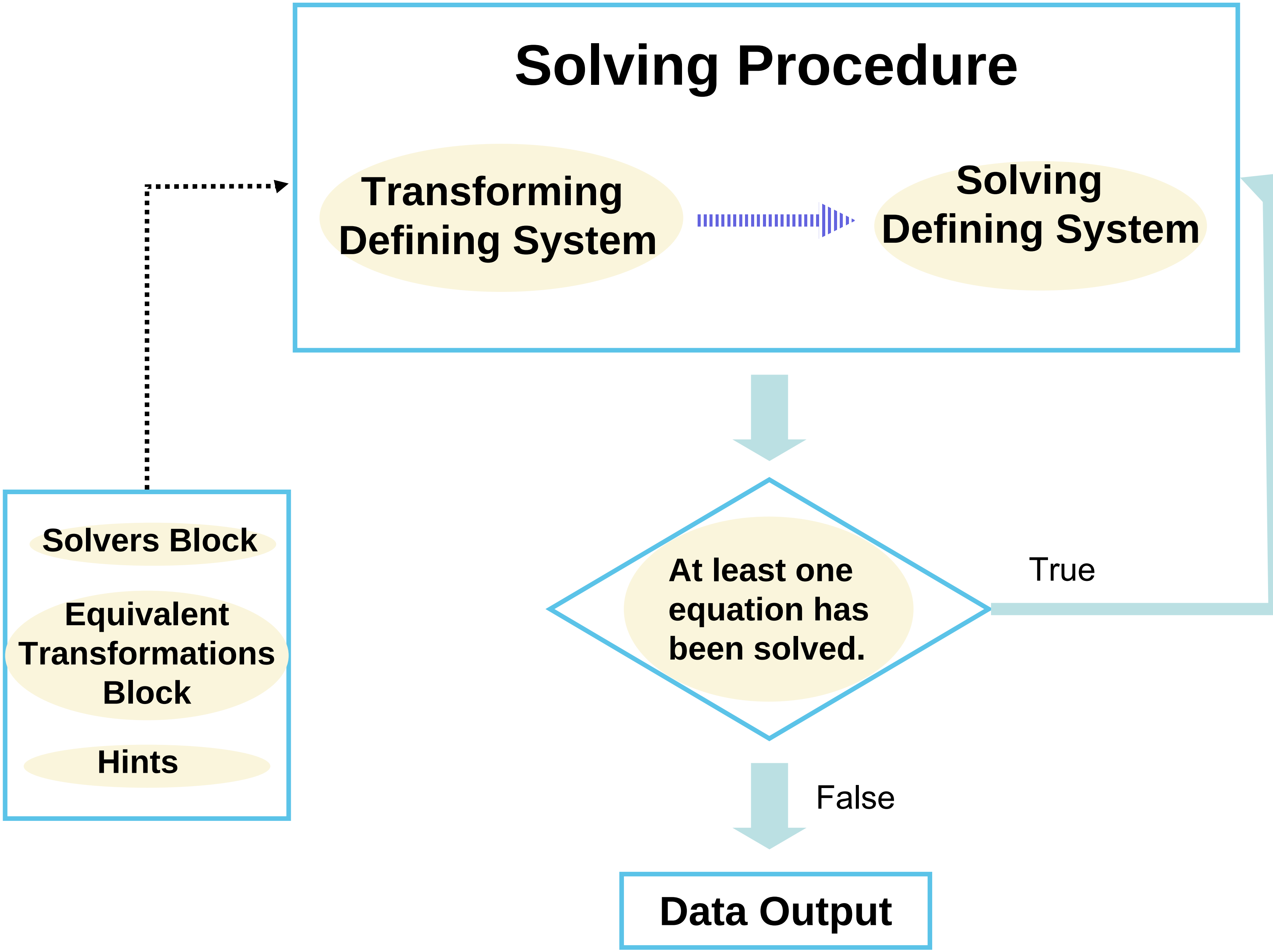
ProlGen $\equiv \mathrm{pr}^n$**(InfGen)**

- $\{\xi^1(x,u), \ldots, \xi^p(x,u), \ldots, \eta^1(x,u), \ldots, \eta^q(x,u)\}$ ***are unknown functions that are to be determined and given at the package output as solutions of the defining system.***

Creating Defining System

Infinitesimal Criterion → **Defining System**

- ***Defining System is the major object in the program.***
- ***Defining System is created by applying the infinitesimal criterion InfGen (LHS)*** $|_{Man}=0$.
- ***Defining System consists of linear partial differential equations.***



Solving Procedure
Transforming Defining System
Solving Defining System
Solvers Block
Equivalent Transformations Block
Hints
At least one equation has been solved.
True
False
Data Output

Equivalent Transformations Block

Module-1 **for adding and subtracting of two equations**

Module-3 **for differentiating of the equations**

Module-4 **for breaking the equations into parts**

- ***The block is open for adding new modules of equivalent transformations.***

Solvers Block

Module-1 — solver of $C_1 x + C_2 = 0$

Module-2 — solver of $C_1 x + C_2 y = 0$

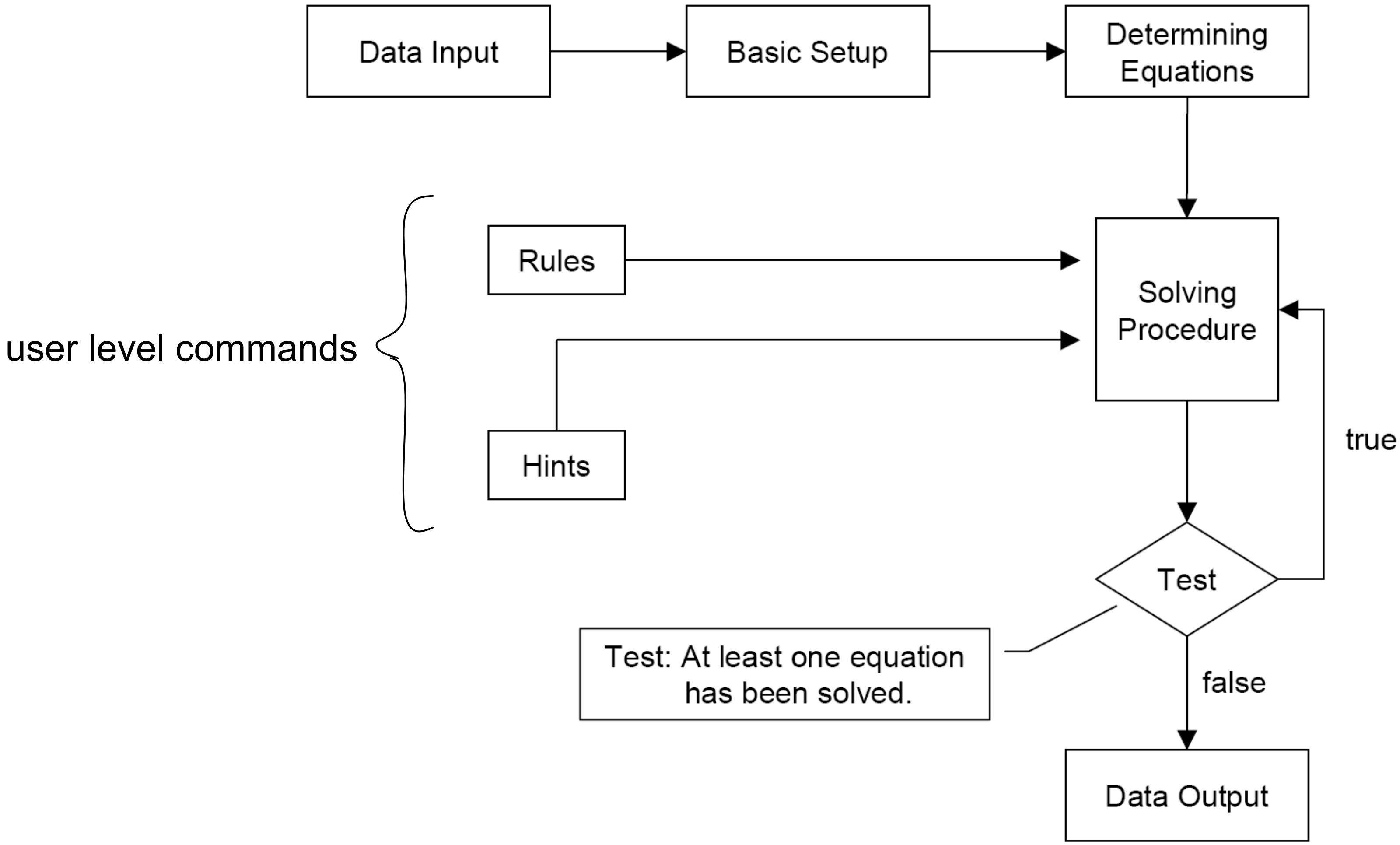
Module-3 — solver of $C_1 y' + C_2 = 0$

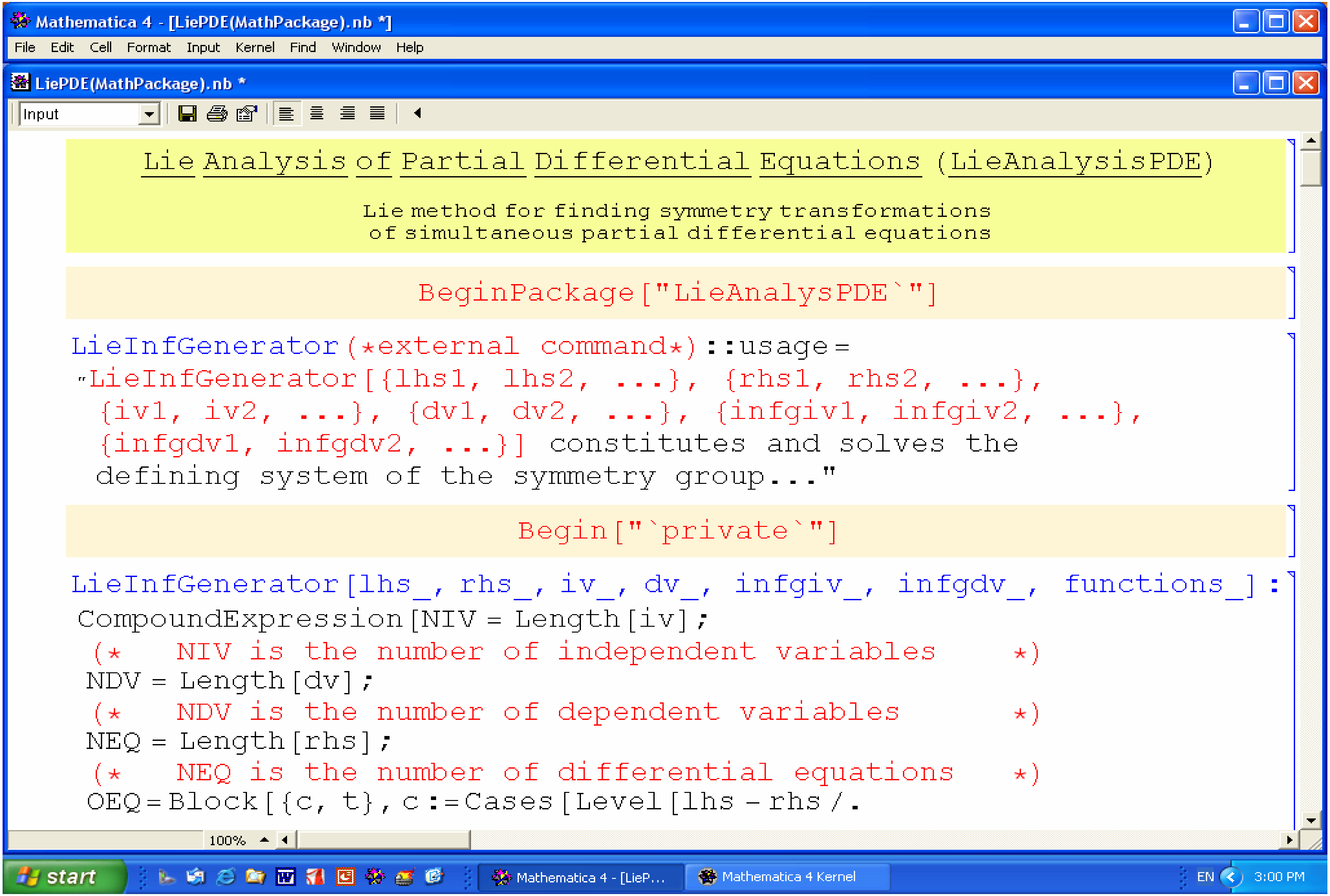
Module-4 — solver of $C_1 y'' + C_2 = 0$

Module-5 — solver of $C_1 y''' + C_2 = 0$

- ***The block is open for adding new modules for solving equations.***

Interactive Mode





Lie Analysis of Partial Differential Equations (LieAnalysisPDE)

Lie method for finding symmetry transformations
of simultaneous partial differential equations

```
BeginPackage["LieAnalysPDE`"]

LieInfGenerator(*external command*)::usage=
"LieInfGenerator[{lhs1, lhs2, ...}, {rhs1, rhs2, ...},
 {iv1, iv2, ...}, {dv1, dv2, ...}, {infgiv1, infgiv2, ...},
 {infgdv1, infgdv2, ...}] constitutes and solves the
 defining system of the symmetry group..."

Begin["`private`"]

LieInfGenerator[lhs_, rhs_, iv_, dv_, infgiv_, infgdv_, functions_]:
 CompoundExpression[NIV = Length[iv];
  (*   NIV is the number of independent variables     *)
  NDV = Length[dv];
  (*   NDV is the number of dependent variables     *)
  NEQ = Length[rhs];
  (*   NEQ is the number of differential equations     *)
  OEQ=Block[{c, t}, c:=Cases[Level[lhs - rhs /.
```

Heat Equation

$$u_t - u_{xx} = 0$$

Input

LieInfGenerator {u[t]}, {u[x, x]}, {x, t}, {u}, { infgenx , infgent }, { infgenu }]

Output

{infgenx $\rightarrow$ c[1] t + c[4] x t + c[5] x + c[2],

infgent $\rightarrow$ c[4] t + c[5] t - c[6] },

{infgenu $\rightarrow$ - c[4] xu - c[4] t u - c[4] x u - c[3] u + f_1[x, t] }

{$f_1^{(0,1)}$ [x, t] - $f_1^{(2,0)}$ [x, t] == 0}

Mathematica 4 - [LiePDE_HeatEqn.nb]

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LiePDE_HeatEqn.nb

Tracing the Evaluation

Heat equation

$$u_t - u_{xx} = 0$$

```
LSS=24
a[[1]]= f[6][x[1], x[2]]
a[[2]]= f[7][x[2]]
a[[3]]= x[3] f[8][x[1], x[2]] + f[9][x[1], x[2]]
SolvingProportFuncEqn: TNF= 9
SolvingLinearEqn0: TNF= 9
SolvingLinearEqn1: TNF= 9
SolvingLinearEqn2: TNF= 9
System[[4]]= 2 f[6]^(2,0)[x[1], x[2]]
a[6]= x[1] f[10][x[2]] + f[11][x[2]]
SolvingLinearEqn3: TNF= 11
LSS=21
a[[1]]= x[1] f[10][x[2]] + f[11][x[2]]
a[[2]]= f[7][x[2]]
a[[3]]= x[3] f[8][x[1], x[2]] + f[9][x[1], x[2]]
SolvingProportFuncEqn: TNF= 11
SolvingLinearEqn0: TNF= 11
SolvingLinearEqn1: TNF= 11
SolvingLinearEqn2: TNF= 11
SolvingLinearEqn3: TNF= 11
System[[16]]= -2 f[8]^(3,0)[x[1], x[2]]
a[8]= 1/2 x[1]^2 f[12][x[2]] + x[1] f[13][x[2]] + f[14][x[2]]
LSS=27
```

$$C_1 x + C_2 y = 0$$

$$C_1 x + C_2 = 0$$

$$C_1 y' + C_2 = 0$$

$$C_1 y'' + C_2 = 0$$

$$C_1 y''' + C_2 = 0$$

100%

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Tracing the Evaluation

Mathematica 4 - [LiePDE_VNLS_h_nu1=nu2=1.nb]

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LiePDE_VNLS_h_nu1=nu2=1.nb

```
LSS=131
a[[1]]= f[31][x[1], x[2], x[6]]
a[[2]]= f[32][x[2], x[5], x[6]]
a[[3]]= x[4] f[33][x[1], x[2], x[6]] + x[3] x[5] f[36][x[1], x[2], x[6]] +
  x[3] f[37][x[1], x[2], x[6]] + x[5] f[38][x[1], x[2], x[6]] + f[39][x[1], x[2], x[6]]
a[[4]]= x[3] f[34][x[1], x[2], x[6]] + x[5] f[35][x[1], x[2], x[4]] + x[4] f[40][x[1], x[2], x[6]] + f[41][x[1], x[2], x[6]]
a[[5]]= x[4] f[27][x[2], x[5], x[6]] + x[3] f[29][x[2], x[5], x[6]] + x[6] f[42][x[1], x[2], x[5]] + f[43][x[1], x[2], x[5]]
a[[6]]= -x[3] f[27][x[2], x[5], x[6]] + x[4] f[29][x[2], x[5], x[6]] + x[5] f[44][x[1], x[2], x[6]] + f[45][x[1], x[2], x[6]]
SolvingProportFuncEqn: TNF= 45
System[[1]]= 4 h f[33][x[1], x[2], x[6]] + 4 h f[34][x[1], x[2], x[6]]
a[33]= -f[34][x[1], x[2], x[6]]
SolvingLinearEqn0: TNF= 45
System[[3]]= 2 f[36][x[1], x[2], x[6]]
a[36]= 0
SolvingLinearEqn1: TNF= 45
System[[64]]= -2 f[27]^(0,1,0)[x[2], x[5], x[6]]
System[[70]]= 2 f[29]^(0,1,0)[x[2], x[5], x[6]]
System[[8]]= -4 f[31]^(0,0,1)[x[1], x[2], x[6]]
System[[73]]= -2 f[32]^(0,1,0)[x[2], x[5], x[6]]
System[[102]]= 2 f[35]^(1,0,0)[x[1], x[2], x[4]]
System[[33]]= -2 f[37]^(0,0,1)[x[1], x[2], x[6]]
System[[104]]= 2 f[38]^(1,0,0)[x[1], x[2], x[6]]
System[[42]]= -2 f[40]^(0,0,1)[x[1], x[2], x[6]]
System[[44]]= 2 f[44]^(0,0,1)[x[1], x[2], x[6]]
```

$C_1 x + C_2 y = 0$

$C_1 x + C_2 = 0$

$C_1 y' + C_2 = 0$

Coupled Nonlinear Schrödinger Equations

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + h|B|^2\right)A = 0$$

$$i\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + h|A|^2\right)B = 0$$

Length of Solved System = 131

100%

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Trial Run

$$X_1 = \partial_x$$

$$X_2 = \partial_t$$

$$X_3 = u\partial_u$$

$$X_4 = x\partial_x + 2t\partial_t$$

$$X_5 = 2t\partial_x - xu\partial_u$$

$$X_6 = 4tx\partial_x + 4t^2\partial_t - \left(x^2 + 2t\right)u\partial_u$$

$$X_\alpha = \alpha\left(x,t\right)\partial_u$$

Heat equation

$$u_t - u_{xx} = 0$$

$\alpha\left(x,t\right)$ **is an arbitrary solution of the Heat Equation**

Trial Run

KdV equation

$$u_t + u_{xxx} + uu_x = 0$$

$X_1 = \partial_x$	**space translation**	$u^{(1)} = f(x-\varepsilon, t)$
$X_2 = \partial_t$	**time translation**	$u^{(2)} = f(x, t-\varepsilon)$
$X_3 = t\partial_x + \partial_u$	**Galilean boost**	$u^{(3)} = f(x-\varepsilon t, t) + \varepsilon$
$X_4 = x\partial_x + 3t\partial_t - 2u\partial_u$	**dilation**	$u^{(4)} = e^{-2\varepsilon} f(e^{-\varepsilon}x, e^{-3\varepsilon}t)$

$u = f(x,t)$ **is an arbitrary solution of the KdV Equation**

$\varepsilon \in R$ **is the group parameter**

References

[1] Schwarz, F., Computing **34** (1985) 91.
[2] Baumann, G., Math. Comp. Simulation **48** (1998) 205.
[3] Baumann, G., Lie Symmetries of Differential equations: a *MATHEMATICA* Program to Determine Lie Symmetries, at www.library.wolfram.com/infocenter/MathSource/431.

Application to Fiber Optics

(physical model)

Coupled Nonlinear Schrödinger Equations (CNSEs)

$$i\frac{\partial A}{\partial x}+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}+\left(|A|^2+\gamma|B|^2-\theta\frac{\partial|A|^2}{\partial t}-\theta\frac{\partial|B|^2}{\partial t}\right)A+\sigma B=0$$

$$i\frac{\partial B}{\partial x}+\frac{\nu}{2}\frac{\partial^2 B}{\partial t^2}+\left(|B|^2+\gamma|A|^2-\theta\frac{\partial|A|^2}{\partial t}-\theta\frac{\partial|B|^2}{\partial t}\right)B+\sigma A=0$$

$\sigma \neq 0$	weak birefringent fibers
$\sigma = 0, \quad \gamma = 2$	two-mode fibers
$\sigma = 0, \quad \gamma = 2/3$	strong birefringent fibers
θ	Raman gain coefficient

Lie Group Analysis

optical fiber

two mode

$\gamma = 2$

strong birefringent

$\gamma = \frac{2}{3}$

group velocity dispersion

negative

$\nu = +1$

positive

$\nu = -1$

Coupled nonlinear Schrödinger equations

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma\,|B|^2\right)A = 0$$

$$i\frac{\partial B}{\partial x} + \frac{\nu}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma\,|A|^2\right)B = 0$$

Admitted Lie point symmetries

$A = z\exp(i\alpha) \quad B = \varsigma\exp(i\beta)$

algebras	$X_1 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial x}$	$X_3 = \frac{\partial}{\partial \alpha}$	$X_4 = \frac{\partial}{\partial \beta}$	$X_5 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial \alpha} + \nu t\frac{\partial}{\partial \beta}$	$X_6 = -t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + \varsigma\frac{\partial}{\partial \varsigma}$
groups	T_1 $t' = t + a_1$	T_2 $x' = x + a_2$	T_3 $\alpha' = \alpha + a_3$	T_4 $\beta' = \beta + a_4$	T_5 $t' = t + a_5 x$ $\alpha' = \alpha + a_5 t + \frac{a_5^2}{2}x$ $\beta' = \beta + \nu a_5 t + \frac{\nu a_5^2}{2}x$	T_6 $t' = t\exp(-a_6)$ $x' = x\exp(-2a_6)$ $z' = z\exp(a_6)$ $\xi' = \xi\exp(a_6)$

Lie Group Analysis

Coupled nonlinear Schrödinger equations

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma |B|^2 - \theta \frac{\partial\left(|A|^2\right)}{\partial t} - \theta \frac{\partial\left(|B|^2\right)}{\partial t} \right) A = 0$$

$$i\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma |A|^2 - \theta \frac{\partial\left(|A|^2\right)}{\partial t} - \theta \frac{\partial\left(|B|^2\right)}{\partial t} \right) B = 0$$

strong birefringent fiber

$$\gamma = \frac{2}{3}$$

strong birefringent fiber
with parallel Raman scattering

$$\theta \neq 0$$

Admitted Lie point symmetries

$A = z\exp(i\alpha)$ $B = \varsigma\exp(i\beta)$

algebras	$X_1 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial x}$	$X_3 = \frac{\partial}{\partial \alpha}$	$X_4 = \frac{\partial}{\partial \beta}$	$X_5 = x\frac{\partial}{\partial t} + t\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)$
groups	T_1: $t' = t + a_1$	T_2: $x' = x + a_2$	T_3: $\alpha' = \alpha + a_3$	T_4: $\beta' = \beta + a_4$	T_5: $t' = t + a_5 x$, $\alpha' = \alpha + a_5 t + \frac{a_5^2}{2}x$, $\beta' = \beta + a_5 t + \frac{a_5^2}{2}x$

Lie Group Analysis

Coupled nonlinear Schrödinger equations

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma\,|B|^2\right)A + kB = 0$$

$$i\frac{\partial B}{\partial x} + \frac{1}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma\,|A|^2\right)B + kA = 0$$

fiber

weak birefringent

$\gamma \neq 0$
$k \neq 0$

nonlinear directional coupler

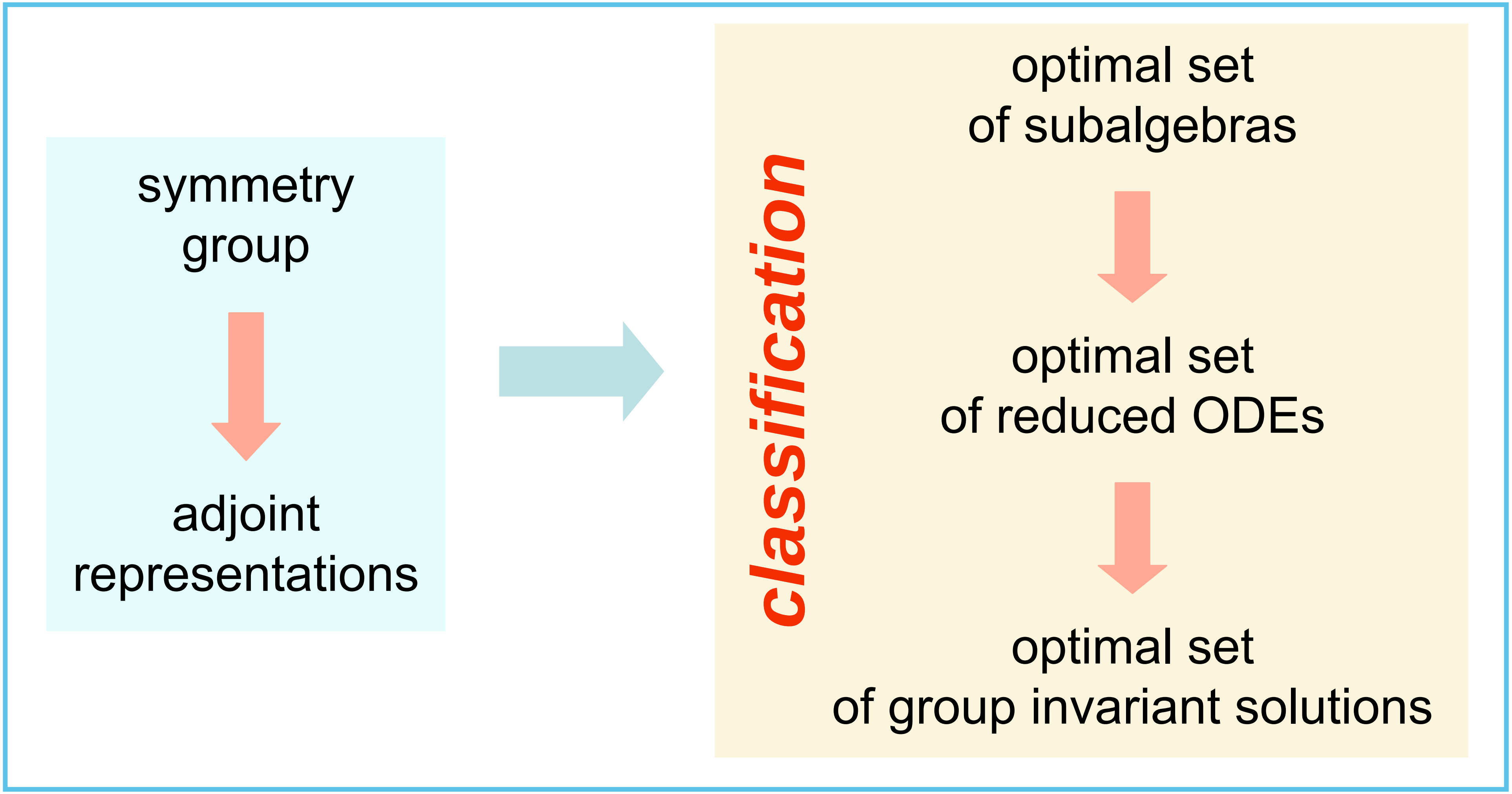
$\gamma = 0$
$k \neq 0$

Admitted Lie point symmetries

$A = z\exp(i\alpha)$ $B = \varsigma\exp(i\beta)$

algebras	$X_1 = \frac{\partial}{\partial t}$	$X_2 = \frac{\partial}{\partial x}$	$X_3 = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha}$	$X_4 = x\frac{\partial}{\partial t} + t\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)$
groups	T_1 $t' = t + a_1$	T_2 $x' = x + a_2$	T_3 $\alpha' = \alpha + a_3$ $\beta' = \beta + a_3$	T_4 $t' = t + a_4 x$ $\alpha' = \alpha + a_4 t + \frac{a_4^2}{2}x$ $\beta' = \beta + a_4 t + \frac{a_4^2}{2}x$

SYMMETRY GROUP REDUCTION



INTERIOR AUTOMORPHISMS

- two mode fibers
- strong birefringent fibers

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma\,|B|^2\right)A = 0$$
$$i\frac{\partial B}{\partial x} + \frac{\nu}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma\,|A|^2\right)B = 0$$

$A_i(\varepsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_1(\varepsilon)$	X_1	X_2	X_3	X_4	$X_5+\varepsilon(X_3+\nu X_4)$	$X_6+\varepsilon X_1$
$A_2(\varepsilon)$	X_1	X_2	X_3	X_4	$X_5-\varepsilon X_1$	$X_6+2\varepsilon X_2$
$A_3(\varepsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_4(\varepsilon)$	X_1	X_2	X_3	X_4	X_5	X_6
$A_5(\varepsilon)$	$X_1+\varepsilon(X_3+\nu X_4)$	$X_2+\varepsilon X_1+\varepsilon(X_3+\nu X_4)$	X_3	X_4	X_5	$X_6-\varepsilon X_5$
$A_6(\varepsilon)$	$e^{-\varepsilon}X_1$	$e^{-2\varepsilon}X_2$	X_3	X_4	$e^{\varepsilon}X_5$	X_6

$$A_i(\varepsilon)X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i,[X_i, X_j]] - \cdots$$

OPTIMAL SET OF SUBALGEBRAS

- two mode fibers
- strong birefringent fibers

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma |B|^2\right)A = 0$$
$$i\frac{\partial B}{\partial x} + \frac{\nu}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma |A|^2\right)B = 0$$

Case A $\quad X_1 + \varepsilon X_3 = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \alpha} \qquad \varepsilon = 0, \pm 1$

Case B $\quad \varepsilon X_4 + X_5 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial \alpha} + (\varepsilon + \nu t)\frac{\partial}{\partial \beta} \qquad \varepsilon = 0, \pm 1$

Case C $\quad X_2 + \delta X_3 + \varepsilon X_4 = \frac{\partial}{\partial x} + \delta\frac{\partial}{\partial \alpha} + \varepsilon\frac{\partial}{\partial \beta} \qquad \varepsilon = 0, \pm 1$ or $\varepsilon = \pm 1, \delta \in R$

Case D $\quad \varepsilon X_2 + \delta X_4 + X_5 = x\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial x} + t\frac{\partial}{\partial \alpha} + (\delta + \nu t)\frac{\partial}{\partial \beta} \qquad \varepsilon = \pm 1, \delta \in R$

Case E $\quad \varepsilon X_3 + \delta X_4 + X_6 = -t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + \varsigma\frac{\partial}{\partial \varsigma} + \varepsilon\frac{\partial}{\partial \alpha} + \delta\frac{\partial}{\partial \beta} \qquad \varepsilon, \delta \in R$

Case F $\quad \varepsilon X_3 + \delta X_4 = \varepsilon\frac{\partial}{\partial \alpha} + \delta\frac{\partial}{\partial \beta} \qquad \varepsilon = 1, \delta = 0$ or $\varepsilon \in R, \delta = 1$

Exact solution for Case A

Nonlinear directional coupler

$$i\frac{\partial A}{\partial x}+\frac{1}{2}\frac{\partial^2 A}{\partial t^2}+|A|^2A+\sigma B=0$$
$$i\frac{\partial B}{\partial x}+\frac{1}{2}\frac{\partial^2 B}{\partial t^2}+|B|^2B+\sigma A=0$$

Reduced system

$$p'+\sigma q\sin(g-f)=0$$
$$q'+\sigma p\sin(f-g)=0$$
$$f'=p^2-\frac{\delta^2}{2}+\sigma\frac{q}{p}\cos(g-f)$$
$$gf=q^2+\sigma\frac{p}{q}\cos(f-g)$$

Exact solution

$$A=\sqrt{\frac{E+E\,\mathrm{cn}\left(2\sigma x\,|\,h^2\right)}{2}}\exp i\left\{\frac{3Ex}{4}-\frac{\arcsin\left(\mathrm{dn}\left(2\sigma x\,|\,h^2\right)\right)}{2}\right\}$$
$$B=\sqrt{\frac{E-E\,\mathrm{cn}\left(2\sigma x\,|\,h^2\right)}{2}}\exp i\left\{\frac{3Ex}{4}-\frac{\arcsin\left(\mathrm{dn}\left(2\sigma x\,|\,h^2\right)\right)}{2}\right\}$$

$$|A|^2+|B|^2=E=\text{const},\quad h=\frac{E}{4\sigma}$$

REDUCTION PROCES

(Case C)

- two mode fibers
- strong birefringent fibers

$$i\frac{\partial A}{\partial x} + \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left(|A|^2 + \gamma|B|^2\right)A = 0$$

$$i\frac{\partial B}{\partial x} + \frac{\nu}{2}\frac{\partial^2 B}{\partial t^2} + \left(|B|^2 + \gamma|A|^2\right)B = 0$$

$A = z\exp(i\alpha) \quad B = \varsigma\exp(i\beta)$

Generator	$X_2 + \delta X_3 + \varepsilon X_4 = \frac{\partial}{\partial x} + \delta\frac{\partial}{\partial \alpha} + \varepsilon\frac{\partial}{\partial \beta}$	$\varepsilon = 0, \pm 1$ or $\varepsilon = \pm 1, \delta \in R$
Invariants	$J_1 = t \quad J_2 = z \quad J_3 = \varsigma \quad J_3 = \alpha - \delta x \quad J_4 = \beta - \varepsilon x$	
New variables	$z = p(x) \quad \varsigma = q(x) \quad \alpha = f(t) + \delta x \quad \beta = g(t) + \varepsilon x$	
Reduced system	$2p'f' + pf'' = 0$ $2q'g' + qg'' = 0$ $p'' - p(f')^2 + 2p^3 + 2\gamma pq^2 - 2\delta p = 0$ $q'' - q(g')^2 + \nu 2q^3 + \nu 2\gamma qp^2 - \nu 2\varepsilon q = 0$	

Exact solution for Case C

(two-mode fibers and strongly birefringent fibers)

$$A = U \exp i \left\{ \frac{C_1}{2\lambda\sqrt{h+1}\, b_1} \Pi(n;\, j \mid m) + \varepsilon x \right\}$$

$$B = U \exp i \left\{ \frac{\pm C_1}{2\lambda\sqrt{h+1}\, b_1} \Pi(n;\, j \mid m) + \varepsilon x \right\}$$

$$\Pi(n;\, j \mid m) = \int_0^j \left[1 - n\, \mathrm{sn}^2 (w \mid m)\right]^{-1} dw$$

$$\mathrm{U} = \sqrt{(b_1 - b_2)\,\mathrm{cn}^2 (j \mid m) + b_2}, \quad j = 2\lambda\sqrt{h+1}\, t, \quad \lambda = \frac{1}{2}\sqrt{b_1 - b_2}$$

$$m = \frac{b_1 - b_2}{b_1 - b_3}, \quad n = \frac{b_1 - b_2}{b_1}, \varepsilon = 0, \pm 1$$

$b_1 > b_2 > b_3$ are the roots of the polynomial $Q(\theta) = \theta^3 - \frac{2\varepsilon}{h+1}\theta^2 - \frac{C_2}{4(h+1)}\theta + \frac{C_1^2}{h+1}$

$\mathrm{sn}(j \mid m)$ and $\mathrm{cn}(j \mid m)$ are the Jacobean sine and cosine elliptic functions

Approximate vector solitary waves

- Strong birefringent fibers with Raman scattering
- A generalized version of previously obtained scalar solitary-wave solution

$$X = \partial_x + cx\,\partial_t + (ct - a)\partial_\alpha + (ct - b)\partial_\beta$$

$$(a - cy)\,p + \frac{1}{2}p_{yy} - \frac{C_1^2}{p^3} + \left(p^2 + hq^2\right)p - 2\theta\left(p^2 p_y + pqq_y\right) = 0$$

$$(b - cy)\,q + \frac{1}{2}q_{yy} - \frac{C_2^2}{q^3} + \left(q^2 + hp^2\right)q - 2\theta\left(q^2 q_y + qpp_y\right) = 0$$

Galilean-like symmetry

reduced system

$$|A| = \sqrt{-2a}\,\mathrm{sech}(z) + \theta\,F(z)\,\mathrm{sech}\,z$$

$$|B| = \theta\,G(z)$$

$$F(z) = -\frac{16a}{15}z + \left(\frac{8a}{15}z^2 - \frac{8a}{5}\ln(\mathrm{sech}\,z)\right)\tanh z$$

$$G_1(z) = \sinh z\,\mathrm{sech}^2 z$$

$$G_2(z) = \mathrm{sech}^2 z$$

$\theta << 1$ – Raman parameter

1. L. Gagnon and P. A. Bélanger, *Soliton self-frequency shift versus Galilean-like symmetry*, Opt. Lett., Vol. 15, No. 9 (1990), pp. 466-468.

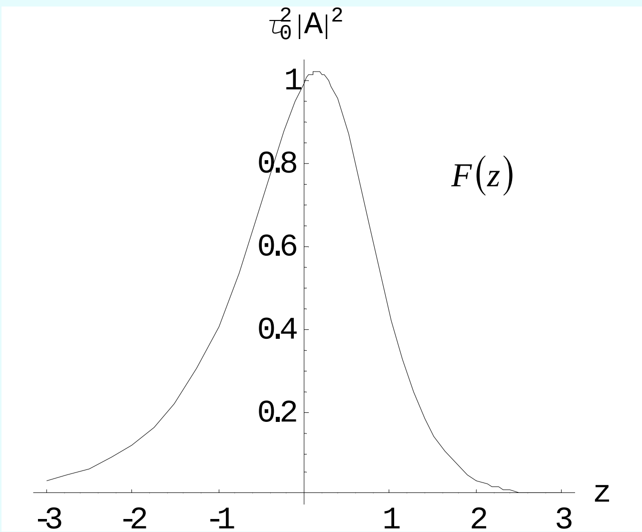
$$|A| = \sqrt{-2a}\,\mathrm{sech}(z) + \theta\, \mathrm{F}(z)\,\mathrm{sech}\, z$$

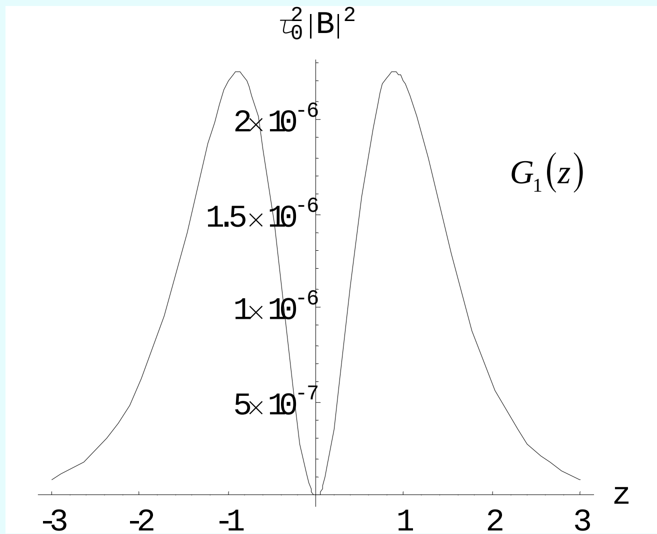
$$|B| = \theta\, G(z)$$

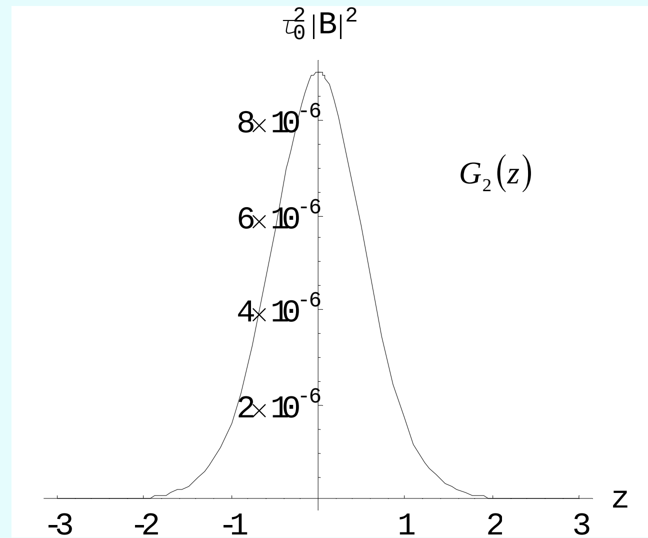
$$F(z) = -\frac{16a}{15} z + \left(\frac{8a}{15} z^2 - \frac{8a}{5} \ln(\mathrm{sech}\, z) \right) \tanh z$$

$$G_1(z) = \sinh z\, \mathrm{sech}^2 z$$

$$G_2(z) = \mathrm{sech}^2 z$$







1. L. Gagnon and P. A. Bélanger, *Soliton self-frequency shift versus Galilean-like symmetry*, Opt. Lett., Vol. 15, No. 9 (1990), pp. 466-468.
2. N. Akhmediev and A. Ankiewicz, *Novel soliton states and bifurcation phenomena in nonlinear fiber couplers*, Phys. Rev. Lett., Vol. 70, No. 16 (1993), pp. 2395-2398.

LAWS OF CONSERVATION

- Two-mode fibers and strong birefringent fibers

SYMMETRY	LAWS OF CONSERVATION
TIME TRANSLATION	$J_1 = \int_{-x}^{x} \left(A_t A^* + B_t B^* \right) dt$
SPACE TRANSLATION	$J_2 \equiv H = \int_{-x}^{x} \left[-\frac{1}{2}\left(\|A_t\|^2 + \nu \|B_t\|^2 \right) + \frac{1}{2}\left(\|A\|^4 + \|B\|^4 \right) + h \|A\|^2 \|B\|^2 \right] dt$
TRANSLATION OF THE PHASE α	$J_3 = \int_{-x}^{x} \|A\|^2 dt$
TRANSLATION OF THE PHASE β	$J_4 = \int_{-x}^{x} \|B\|^2 dt$
GALILEAN-LIKE SYMMETRY	$J_5 = \int_{-x}^{x} t \left(\|A\|^2 + \nu \|B\|^2 \right) dt + ixJ_1$

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Conclusion

- **The symbolic computational tools of *MATHEMATICA* have been applied to determining the Lie symmetries of PDE.**

- **An algorithm for creating and solving the defining system of the symmetry transformations has been developed and implemented in *MATHEMATICA* package.**

- **The package has been successfully applied to basic physical equations from nonlinear fiber optics.**

- **Future work: The package capabilities can be extended by adding new programming modules for transforming and solving other wider classes of differential equations.**