Geometry, Integrability and Quantization
8-13 June 2007
Bulgarian Akademy of Sciences

# Banach Lie-Poisson spaces and integrable systems

Anatol Odzijewicz

Institute of Mathematics
University in Białystok
Lipowa 41
Białystok, Poland
aodzijew@uwb.edu.pl

## References


[1] A. Odzijewicz and T. S. Ratiu. Banach Lie–Poisson spaces and reduction. *Comm. Math. Phys.* 243, 1-54, 2003.

[2] A. Odzijewicz and T. S. Ratiu. Extensions of Banach Lie-Poisson spaces. *J. Funct. Anal.* 217, 103-125, 2004.

[3] A. Odzijewicz and T. S. Ratiu. The Banach Poisson geometry of multi-diagonal Toda lattices. *arxiv.org/abs/math/0310318*

# Banach Lie-Poisson spaces

**Definition 1.** A **Banach Lie algebra** $(\mathfrak{g}, [\cdot,\cdot])$ is a Banach space imposed in the continuous Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

For $x \in \mathfrak{g}$ one defines the adjoint $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\operatorname{ad}_x g := [x, y]$, and coadjoint $\operatorname{ad}_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$ map which are also continuous.

**Definition 2.** A **Banach Lie-Poisson space** $(\mathfrak{b}, \{\cdot,\cdot\})$ is a real or complex Poisson manifold such that $\mathfrak{b}$ is a Banach space and its dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation.

**Theorem 3.** *The Banach space $\mathfrak{b}$ is a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ if and only if it is predual $\mathfrak{b}^* = \mathfrak{g}$ of some Banach Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ satisfying $\operatorname{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}^*$ for all $x \in \mathfrak{g}$. The Poisson bracket of $f, g \in C^\infty(\mathfrak{b})$ is given by*

$$\{f, g\}(b) = \langle [Df(b), Dg(b)]; b\rangle,$$

*where $b \in \mathfrak{b}$.*

- A **morphism** between two Banach Lie-Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ we assume a continuous linear map $\Phi : \mathfrak{b}_1 \longrightarrow \mathfrak{b}_2$ that preserves the linear Poisson structure, i.e.

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$. It will be called a **linear Poisson map**.

- We present Hamilton equation in the form

$$\frac{d}{dt} b = -\operatorname{ad}^*_{Dh(b)} b, \qquad b \in \mathfrak{b},$$

where $h \in C^\infty(\mathfrak{b})$ is a Hamiltonian of the system.

**Example 1.** $L^\infty(\mathcal{H})$ — $C^*$-algebra of the **bounded operators** acting in $\mathcal{H}$.

• One has

$$L^\infty(\mathcal{H}) = (L^1(\mathcal{H}))^*$$

where the duality is given by

$$\langle X; \rho \rangle := \mathrm{Tr}(X\rho),$$

for $\rho \in L^1(\mathcal{H}) := \{\rho \in L^\infty(\mathcal{H}) : \|\rho\|_1 := \mathrm{Tr}\sqrt{\rho^*\rho} < \infty\}$, $X \in L^\infty(\mathcal{H})$.

• The associative Banach algebra $L^\infty(\mathcal{H})$ can be considered as the Banach Lie algebra of the complex Banach Lie group $GL^\infty(\mathcal{H})$ of the invertible elements in $L^\infty(\mathcal{H})$.

- The predual of real Banach Lie algebra

$$U^\infty(\mathcal{H}) := \{X \in L^\infty(\mathcal{H}) : X^* + X = 0\}$$

is

$$U^1(\mathcal{H}) := \{\rho \in L^1(\mathcal{H}) : \rho^* = \rho\}$$

and the isomorphism $U^1(\mathcal{H})^* \cong U^\infty(\mathcal{H})$ is given by

$$\langle X; \rho \rangle := i \operatorname{Tr}(X\rho).$$

- The formula

$$\operatorname{ad}^*_X \rho = [\rho, X],$$

shows that $U^1(\mathcal{H}) \subset U^\infty(\mathcal{H})^*$ is invariant with respect to the coadjoint action of $U^\infty(\mathcal{H})$ on $U^\infty(\mathcal{H})^*$. The above allows us to define Poisson bracket

$$\{F, G\}_{U^1}(\rho) := i \operatorname{Tr}\left(\rho[DF(\rho), DG(\rho)]\right)$$

for $F, G \in C^\infty(U^1(\mathcal{H}))$.

• The Hamilton equations

$$-i\frac{d}{dt}\rho(t) = [\rho(t), DH(\rho(t))],$$

is the non-linear version of the **Liouville-von Neumann equation**.

• One obtains the Liouville-von Neumann equation taking the Hamiltonian $H(\rho) = \mathrm{Tr}(\rho\hat{H})$, where $\hat{H} \in iU^{\infty}(\mathcal{H})$.

$\diamondsuit$

# Induced Banach Lie-Poisson spaces

- $\mathfrak{b}_1$ — a Banach space,
- $(\mathfrak{b}, \{\cdot,\cdot\})$ — a Banach Lie-Poisson space
- $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ an injective continuous linear map with closed range.
- $\ker \iota^*$ is an ideal in $(\mathfrak{b}^*, [\cdot,\cdot]) \Longleftrightarrow \mathfrak{b}_1$ carries a unique Banach Lie-Poisson bracket $\{\cdot,\cdot\}_1^{\mathrm{ind}}$ such that

$$\{F \circ \iota, G \circ \iota\}_1^{\mathrm{ind}} = \{F, G\} \circ \iota$$

for any $F, G \in C^\infty(\mathfrak{b})$. This Poisson structure on $\mathfrak{b}_1$ is said to be **induced** by the mapping $\iota$.

Assume that there exists a projector $R = R^2 : \mathfrak{b} \longrightarrow \mathfrak{b}$ such that $\iota(\mathfrak{b}_1) = R(\mathfrak{b})$. We get

$$\{f, g\}_1^{\mathrm{ind}}(b_1) =$$

$$= \left\langle \left[ D(f \circ \iota^{-1} \circ R)(\iota(b_1)), D(g \circ \iota^{-1} \circ R)(\iota(b_1)) \right], \iota(b_1) \right\rangle .$$

**Proposition 4.** *Let* $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ *be a quasi-immersion of Banach Lie-Poisson spaces* (*so* $\operatorname{range}\iota$ *is a closed subspace of* $\mathfrak{b}$ *and* $\ker\iota^*$ *is an ideal in the Banach Lie algebra* $\mathfrak{b}^*$). *Assume that there is a connected Banach Lie group* $G$ *with Banach Lie algebra* $\mathfrak{g} := \mathfrak{b}^*$. *Then the* $G$*-coadjoint orbit* $\mathcal{O}_{\iota(b_1)} := \operatorname{Ad}^*_G \iota(b_1)$ *is contained in* $\iota(\mathfrak{b}_1)$ *for any* $b_1 \in \mathfrak{b}_1$. *In addition, if* $N \subset G$ *is a closed connected normal Lie subgroup of* $G$ *whose Lie algebra is* $\ker\iota^*$, *then the* $N$*-coadjoint action restricted to* $\iota(\mathfrak{b}_1)$ *is trivial.*
*Therefore the Banach Lie group* $G/N := \{[g] := gN \mid g \in G\}$ *naturally acts on* $\iota(\mathfrak{b}_1)$ *and the orbit of* $\iota(b_1)$ *under this action coincides with* $\mathcal{O}_{\iota(b_1)}$ *for any* $b_1 \in \mathfrak{b}_1$.

# Coinduced Banach Lie-Poisson spaces

- $(\mathfrak{b}, \{\ ,\ \})$ — a Banach Lie-Poisson space
- $\mathfrak{b}_1$ — the Banach space
- $\pi : \mathfrak{b} \to \mathfrak{b}_1$ — a continuous linear surjective map

- $\pi^*(\mathfrak{b}_1^*) \subset \mathfrak{b}^*$ is closed under the Lie bracket $[\cdot\,,\cdot]$ of $\mathfrak{b}^* \Longleftrightarrow \mathfrak{b}_1$ carries a unique Banach Lie-Poisson bracket $\{\ ,\ \}_1^{\mathrm{coind}}$ such that

$$\{f \circ \pi, g \circ \pi\} = \{f, g\}_1^{\mathrm{coind}} \circ \pi$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$. This unique Poisson structure on $\mathfrak{b}_1$ is said to be **coinduced** by map $\pi$.

- The coinduced bracket has then the form

$$\{f, g\}_1^{\mathrm{coind}}(b_1) = \left\langle (\pi^*)^{-1} \left[\pi^*(Df(b_1)), \pi^*(Dg(b_1))\right], b_1 \right\rangle$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$ and $b_1 \in \mathfrak{b}_1$.

• Let us assume splitting $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ .

• $R_j : \mathfrak{b} \to \mathfrak{b}$ the projection onto $\mathfrak{b}_j$, for $j = 1, 2$.

• Dual projectors $R_1^*, R_2^* : \mathfrak{b}^* \to \mathfrak{b}^*$

One has

$\ker R_1 = \operatorname{im} R_2 = \mathfrak{b}_2$
$\ker R_2 = \operatorname{im} R_1 = \mathfrak{b}_1$
$\ker R_1^* = \operatorname{im} R_2^* = (\operatorname{im} R_1)^\circ \cong \mathfrak{b}_2^*$
$\ker R_2^* = \operatorname{im} R_1^* = (\operatorname{im} R_2)^\circ \cong \mathfrak{b}_1^*$
$\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$
$\mathfrak{b}^* = \mathfrak{b}_2^\circ \oplus \mathfrak{b}_1^\circ \cong \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*$.

• The splitting determines the maps

$$\begin{tikzcd}
 & \mathfrak{b} \arrow[dl, shift right, "\pi_1"'] \arrow[dr, shift left, "\pi_2"] & \\
\mathfrak{b}_1 \arrow[ur, shift right, "\iota_1"'] & & \mathfrak{b}_2 \arrow[ul, shift left, "\iota_2"]
\end{tikzcd}$$

and

$$\{f, g\}_j^{\text{coind}}(b_j) = \tag{0.1}$$
$$= \langle [D(f \circ \pi_j)(\iota_j(b_j)), D(g \circ \pi_j)(\iota_j(b_j))] \, , \iota_j(b_j) \rangle \, ,$$

where $\quad b_j \in \mathfrak{b}_j$.

**Proposition 5.** *Assume that* $\operatorname{im} R_1^*$ *and* $\operatorname{im} R_2^*$ *are Banach Lie subalgebras of* $\mathfrak{b}^*$. *Then:*

**(i)** $\mathfrak{b}_j$ *has a Banach Lie-Poisson structure coinduced by* $\pi_j$ *and the expression of the coinduced bracket* $\{\,,\}_j^{\mathrm{coind}}$ *on* $\mathfrak{b}_j$ *is given by* (0.15). *The Hamiltonian vector field of* $h \in C^\infty(\mathfrak{b}_j)$ *at* $b_j \in \mathfrak{b}_j$ *is given by*

$$X_h(b_j) = -\pi_j \left( \mathrm{ad}^*_{\pi_j^* Dh(b_j)} \iota_j(b_j) \right), \qquad j = 1, 2, \tag{0.2}$$

*where* $Dh(b_j) \in \mathfrak{b}_j^*$ *and* $\mathrm{ad}_x$ *is the adjoint action of* $x \in \mathfrak{b}^*$ *on* $\mathfrak{b}^*$.

**(ii)** *The Banach space isomorphism* $R := \frac{1}{2}(R_1 - R_2) : \mathfrak{b} \to \mathfrak{b}$ *defines a new Banach Lie-Poisson structure*

$$\{f, g\}_R(b) := \tag{0.3}$$

$$= \langle [R^* Df(b), Dg(b)] + [Df(b), R^* Dg(b)], b \rangle$$

*on* $\mathfrak{b}$, $f, g \in C^\infty(\mathfrak{b})$, *that coincides with the product structure on* $\mathfrak{b}_1 \times \overline{\mathfrak{b}}_2$, *where* $\mathfrak{b}_1$ *carries the coinduced bracket* $\{\,,\}_1^{\mathrm{coind}}$

*and* $\overline{\mathfrak{b}}_2$ *denotes* $\mathfrak{b}_2$ *endowed with the Lie-Poisson bracket* $-\{\,,\}_2^{\mathrm{coind}}$.

**(iii)** *The inclusion maps* $\iota_1 : (\mathfrak{b}_1, \{\,,\}_1^{\mathrm{coind}}) \hookrightarrow (\mathfrak{b}, \{\,,\}_R)$ *and* $\iota_2 : (\overline{\mathfrak{b}}_2, \{\,,\}_2^{\mathrm{coind}}) \hookrightarrow (\mathfrak{b}, \{\,,\}_R)$ *are linear injective Poisson maps with closed range.*

**(iv)** *The map* $\iota_j$ *induces from* $(\mathfrak{b}, \{\,,\}_R)$ *a Banach Lie-Poisson structure on* $\mathfrak{b}_j$ *which coincides with the coinduced structure described in* **(i)**, *for* $j = 1, 2$.

**Corrolary 6** (Involution Theorem)**.** *In the notations and hypotheses of Proposition 5 we have:*

**(i)** *The Casimir functions on* $(\mathfrak{b}, \{\cdot,\cdot\})$ *are in involution on* $(\mathfrak{b}, \{\cdot,\cdot\}_R)$ *and restrict to functions in involution on* $\mathfrak{b}_j$*,* $j = 1, 2$*.*

**(ii)** *If* $H$ *is a Casimir function on* $\mathfrak{b}$*, then its restriction* $H \circ \iota_j$ *to* $\mathfrak{b}_j$ *has the Hamiltonian vector field*

$$\begin{aligned} X_{H\circ\iota_1}(b_1) &= \pi_1\big(\operatorname{ad}^*_{R_2^* DH(\iota_1(b_1))}\,\iota_1(b_1)\big) \\ X_{H\circ\iota_2}(b_2) &= \pi_2\big(\operatorname{ad}^*_{R_1^* DH(\iota_2(b_2))}\,\iota_2(b_2)\big) \end{aligned} \tag{0.4}$$

*for any* $b_1 \in \mathfrak{b}_1$ *and* $b_2 \in \mathfrak{b}_2$*, where* $\iota_j : \mathfrak{b}_j \hookrightarrow \mathfrak{b}$ *is the inclusion,* $j = 1, 2$*.*

Taken together, **Proposition 5** and **Corollary 6** give a version of the **Adler-Kostant-Symes Theorem** formulated with the necessary additional hypotheses in the context of Banach Lie-Poisson spaces.

**Proposition 7.** *Let $(\mathfrak{b}, \{\, ,\})$ be a Banach Lie-Poisson space and let $R_1, R_3 : \mathfrak{b} \to \mathfrak{b}$ be projectors. Assume that $\operatorname{im} R_{21} = \operatorname{im} R_{23} =: \mathfrak{b}_2$, where $R_{21} := \mathrm{id}_{\mathfrak{b}} - R_1$, $R_{23} := \mathrm{id}_{\mathfrak{b}} - R_3$, and denote $\mathfrak{b}_1 := \operatorname{im} R_1$, $\mathfrak{b}_3 := \operatorname{im} R_3$. We summarize this situation in the diagram*

$$\begin{tikzcd} & \mathfrak{b} \arrow[dl, shift right, "\pi_1"'] \arrow[dr, "\pi_{21}"] & & \mathfrak{b} \arrow[dl, "\pi_{23}"'] \arrow[dr, shift left, "\pi_3"] & \\ \mathfrak{b}_1 \arrow[ur, shift right, "\iota_1"'] & & \mathfrak{b}_2 & & \mathfrak{b}_3 \arrow[ul, shift left, "\iota_3"] \end{tikzcd}$$

*where $\pi_1, \pi_{21}, \pi_{23}, \pi_3$ are the projections onto the ranges of $R_1, R_{21}, R_{23}$, and $R_3$ respectively, according to the splittings $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 = \mathfrak{b}_2 \oplus \mathfrak{b}_3$, and $\iota_1 : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$, $\iota_3 : \mathfrak{b}_3 \hookrightarrow \mathfrak{b}$ are the inclusions.*

*Then one has:*

**(i)** *If $\mathfrak{b}_2^\circ$ is a Banach Lie subalgebra of $\mathfrak{b}^*$, then $\Phi_{31} := \pi_3 \circ \iota_1 : (\mathfrak{b}_1, \{\, ,\}_1^{\mathrm{coind}}) \to (\mathfrak{b}_3, \{\, ,\}_3^{\mathrm{coind}})$ and $\Phi_{13} := \pi_1 \circ \iota_3 : (\mathfrak{b}_3, \{\, ,\}_3^{\mathrm{coind}}) \to (\mathfrak{b}_1, \{\, ,\}_1^{\mathrm{coind}})$ are mutually inverse linear Poisson isomorphisms.*

**(ii)** *If* $\mathfrak{b}_1^\circ$ *and* $\mathfrak{b}_3^\circ$ *are Banach Lie subalgebras of* $\mathfrak{b}^*$*, then* $\mathfrak{b}_2$ *has two coinduced Banach Lie-Poisson brackets* $\{\,,\}_{21}^{\mathrm{coind}}$ *and* $\{\,,\}_{23}^{\mathrm{coind}}$ *which are not isomorphic in general.*

# Induction and coinduction from $L^1(\mathcal{H})$

- Since $L^1(\mathcal{H}) \subset L^2(\mathcal{H})$, where

$L^2(\mathcal{H}) := \{\rho \in L^\infty(\mathcal{H}) : \|\rho\|_2 := \sqrt{\operatorname{Tr} \rho^* \rho} < \infty\}$

is the ideal of **Hilbert-Schmidt operators** in $\mathcal{H}$, one can consider

$$\Big\{ |m\rangle\langle n| \Big\}_{n,m=0}^{\infty}$$

as **Schauder basis** of $L^1(\mathcal{H})$. The biorthogonal functionals

$$\Big\{ \operatorname{Tr}(|k\rangle\langle l| \cdot \,) \Big\}_{k,l=0}^{\infty}$$

form the basis of $L^\infty(\mathcal{H})$ in sense of the weak*-topology.

• We assume that $\mathcal{H}$ is the real separable Hilbert space; $L^\infty := L^\infty(\mathcal{H})$, $L^1 := L^1(\mathcal{H})$ and define the shift operator:

$$S := \sum_{n=0}^{\infty} |n\rangle\langle n+1|$$

$$s(x_0, x_1, x_2 \ldots, x_n, \ldots) := (x_1, x_2, \ldots, x_n, \ldots)$$

for any

$$(x_0, x_1, x_2 \ldots, x_n, \ldots) \in \ell^\infty \cong L_0^\infty,$$

where $L_0^\infty$ and $L_0^1$ are diagonal parts of $L^\infty$ and $L^1$ respectively.

Any $x \in L^\infty$ and $\rho \in L^1$ can be written as

$$x = \sum_{j=1}^{\infty} (S^T)^j x_{-j} + x_0 + \sum_{i=1}^{\infty} x_i S^i, \qquad (0.5)$$

$$\rho = \sum_{j=1}^{\infty} (S^T)^j \rho_j + \rho_0 + \sum_{i=1}^{\infty} \rho_{-i} S^i, \qquad (0.6)$$

where $x_i, x_0, x_{-j} \in L_0^\infty$ and $\rho_j, \rho_0, \rho_{-i} \in L_0^1$.

We have decompositions

$$L^\infty = \bigoplus_{k\in\mathbb{Z}} L^\infty_k \qquad \text{and} \qquad L^1 = \bigoplus_{k\in\mathbb{Z}} L^1_k.$$

where

$$L^\infty_k := \{\rho \in L^\infty \mid \rho_{nm} = 0 \text{ for } m \neq n+k\} \subset L^\infty$$
$$L^1_k := \{\rho \in L^1 \mid \rho_{nm} = 0 \text{ for } m \neq n+k\} \subset L^1$$

Banach subspaces of $L^1(\mathcal{H})$

- $L^1_- := \oplus_{k=-\infty}^{0} L^1_k$ and $L^1_+ := \oplus_{k=0}^{\infty} L^1_k$
- $L^1_S := \{\rho \in L^1 \mid \rho = \rho^T\}$ and $L^1_A := \{\rho \in L^1 \mid \rho = -\rho^T\}$
- $L^1_{-,k} := \oplus_{i=-k+1}^{0} L^1_i$ and $L^1_{+,k} := \oplus_{i=0}^{k-1} L^1_i$, for $k \geqslant 1$
- $I^1_{-,k} := \oplus_{i=-\infty}^{-k} L^1_i$ and $I^1_{+,k} := \oplus_{i=k}^{\infty} L^1_i$, for $k \geqslant 1$
- $L^1_{S,k} := L^1_S \cap \left(L^1_{+,k} + L^1_{-,k}\right)$ and $L^1_{A,k} := L^1_A \cap \left(L^1_{+,k} + L^1_{-,k}\right)$, for $k \geqslant 1$.

and Banach subspaces $L^\infty(\mathcal{H})$

- $L^\infty_- := \oplus_{k=-\infty}^{0} L^\infty_k$ and $L^\infty_+ := \oplus_{k=0}^{\infty} L^\infty_k$
- $L^\infty_S := \{x \in L^\infty \mid x^T = x\}$ and $L^\infty_A := \{x \in L^\infty \mid x^T = -x\}$
- $L^\infty_{-,k} := \oplus_{i=-k+1}^{0} L^\infty_i$ and $L^\infty_{+,k} := \oplus_{i=0}^{k-1} L^\infty_i$, for $k \geqslant 1$
- $I^\infty_{-,k} := \oplus_{i=-\infty}^{-k} L^\infty_i$ and $I^\infty_{+,k} := \oplus_{i=k}^{\infty} L^\infty_i$, for $k \geqslant 1$

- $L^{\infty}_{S,k} := L^{\infty}_{S} \cap \left(L^{\infty}_{+,k} + L^{\infty}_{-,k}\right)$ and $L^{\infty}_{A,k} := L^{\infty}_{A} \cap \left(L^{\infty}_{+,k} + L^{\infty}_{-,k}\right)$, for $k \geqslant 1$.

Splittings of Banach spaces

$$L^1 = L^1_- \oplus I^1_{+,1}, \qquad L^1 = L^1_S \oplus I^1_{+,1},$$

$$L^1_- = L^1_{-,k} \oplus I^1_{-,k}, \qquad L^\infty = L^\infty_+ \oplus I^\infty_{-,1}, \quad (0.7)$$

$$L^\infty = L^\infty_+ \oplus L^\infty_A, \qquad L^\infty_+ = L^\infty_{+,k} \oplus I^\infty_{+,k}$$

are related by

$$\begin{array}{ll} (L^1_-)^* \cong (I^1_{+,1})^\circ = L^\infty_+, & (L^1_S)^* \cong (I^1_{+,1})^\circ = L^\infty_+ \\ (L^1_{-,k})^* \cong (I^1_{-,k})^\circ = L^\infty_{+,k}, & (I^1_{+,1})^* \cong (L^1_-)^\circ = I^\infty_{-,1} \\ (I^1_{+,1})^* \cong (L^1_S)^\circ = L^\infty_A, & (I^1_{-,k})^* \cong (L^1_{-,k})^\circ = I^\infty_{+,k} \end{array}$$

• Taking the maps defined by the splittings (0.7)

$$\begin{tikzcd}
 & L^1 \arrow[dl, shift right, "\pi_S"'] \arrow[dr, shift left, "\pi_{S,+}"] & & L^1 \arrow[dl, shift right, "\pi_+"'] \arrow[dr, shift left, "\pi_-"] & \\
L^1_S \arrow[ur, shift right, "\iota_S"'] & & I^1_{+,1} \arrow[ul, shift left, "\iota_{S,+}"] \arrow[ur, shift right, "\iota_+"'] & & L^1_- \arrow[ul, shift left, "\iota_-"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
L^1_- \arrow[r, "\Phi_{S,-}"] \arrow[d, shift left, "\pi_{-,k}"] & L^1_S \arrow[d, shift left, "\pi_{S,k}"] \\
L^1_{-,k} \arrow[r, "\Phi_{S,-,k}"] \arrow[u, shift left, "\iota_{-,k}"] & L^1_{S,k} \arrow[u, shift left, "\iota_{S,k}"]
\end{tikzcd}$$

where $\Phi_{S,-} := \pi_S \circ \iota_- : L^1_- \to L^1_S$ and $\Phi_{S,-,k} := \pi_{S,k} \circ \Phi_{S,-} \circ \iota_{-,k} : L^1_{-,k} \to L^1_{S,k}$, and using the induction and coinduction procedures we obtain the Banach Lie-Poisson structure on $L^1_{-,k}$ with the Poisson bracket given by

$$\{f,g\}_k(\rho) = \mathrm{Tr}\left(\rho\left[Df(\rho), Dg(\rho)\right]_k\right) = \tag{0.8}$$

$$= \sum_{l=0}^{k-1} \sum_{i=0}^{l} \mathrm{Tr}\left[\rho_l\left(\frac{\delta f}{\delta \rho_i}(\rho) s^i\left(\frac{\delta g}{\delta \rho_{l-i}}(\rho)\right) - \right.\right.$$

$$\left.\left. - \frac{\delta g}{\delta \rho_i}(\rho) s^i\left(\frac{\delta f}{\delta \rho_{l-i}}(\rho)\right)\right)\right]$$

for $f, g \in C^\infty(L^1_{-,k})$, where $\frac{\delta f}{\delta \rho_i}(\rho)$ denotes the partial functional derivative of $f$ relative to $\rho_i$ defined by $Df(\rho) = \frac{\delta f}{\delta \rho_0}(\rho) + \frac{\delta f}{\delta \rho_1}(\rho) S + \cdots + \frac{\delta f}{\delta \rho_{k-1}}(\rho) S^{k-1}$.

• If in the previous formulas we let $k = \infty$ one obtains the Lie-Poisson bracket on $L^1_-$.

• Banach Lie-Poisson spaces $L^1_S$ and $L^1_-$ are isomorphic.

• $I^1_{+,1}$ is the predual of the two Banach Lie algebras $I^\infty_{-,1}$ and $L^\infty_A$ thus it carries two different Lie-Poisson brackets:

$$\{f, g\}_+(\rho) = \tag{0.9}$$

$$= \operatorname{Tr}\left(\iota_+(\rho)\left[D(f \circ \pi_+)(\iota_+(\rho)), D(g \circ \pi_+)(\iota_+(\rho))\right]\right)$$

and

$$\{f, g\}_{S,+}(\rho) = \tag{0.10}$$

$$\operatorname{Tr}\left(\iota_{S+}(\rho)\left[D(f \circ \pi_{S,+})(\iota_{S,+}(\rho)), D(g \circ \pi_{S,+})(\iota_{S,+}(\rho))\right]\right),$$

where $\rho \in I^1_{+,1}$, $f, g \in C^\infty(I^1_{+,1})$.

• One has the $GL^\infty_+$-invariant filtrations

$$\iota_{-,1}(L^1_{-,1}) \hookrightarrow \iota_{-,2}(L^1_{-,2}) \hookrightarrow \ldots \hookrightarrow \iota_{-,k}(L^1_{-,k}) \hookrightarrow \hookrightarrow \iota_{-,k+1}(L^1_{-,k+1}) \hookrightarrow \ldots \hookrightarrow L^1_- \tag{0.11}$$

$$\iota_{S,1}(L^1_{S,1}) \hookrightarrow \iota_{S,2}(L^1_{S,2}) \hookrightarrow \ldots \hookrightarrow \iota_{S,k}(L^1_{S,k}) \hookrightarrow \hookrightarrow \iota_{S,k+1}(L^1_{S,k+1}) \hookrightarrow \ldots \hookrightarrow L^1_S \tag{0.12}$$

of Banach Lie-Poisson spaces predual to the sequence

$$L^\infty_+ \longrightarrow \ldots \longrightarrow L^\infty_{+,k} \longrightarrow L^\infty_{+,k-1} \longrightarrow \ldots \ldots \longrightarrow L^\infty_{+,2} \longrightarrow L^\infty_{+,1} \tag{0.13}$$

of Banach Lie algebras in which each arrow is the surjective projector $\pi^\infty_{+,k,k-1} : L^\infty_{+,k} \to L^\infty_{+,k-1}$ that maps $k$-diagonal upper triangular operators to $(k-1)$-diagonal upper triangular operators by eliminating the $k$-diagonal. We have $\pi^\infty_{+,k,k-1} \circ \pi^\infty_{+,k} = \pi^\infty_{+,k-1}$. The Banach Lie algebra structure on $L^\infty_{+,k}$ is given by the isomorphism $L^{\infty}_{+,k} \cong L^{\infty}_{+}/I^{\infty}_{+,k}$.

• A $k$**-diagonal Hamiltonian system** is, by definition, a Hamiltonian system on the Banach Lie-Poisson space

$$\left(L^1_{-,k}, \{\cdot,\cdot\}_k\right) \xrightarrow{\sim} \left(L^1_{S,k}, \{\cdot,\cdot\}_{S,k}\right).$$

• Hamilton's equations on $\left(L^1_{-,k}, \{\cdot,\cdot\}_k\right)$ for Hamiltonians $h_k \in C^\infty(L^1_{-,k})$ are given by

$$\frac{d}{dt}\rho_j = -\sum_{l=j}^{k-1}\left(\tilde{s}^{l-j}\left(\rho_l\frac{\delta h_k}{\delta\rho_{l-j}}\right) - \rho_l s^j\left(\frac{\delta h_k}{\delta\rho_{l-j}}\right)\right) \tag{0.14}$$

for $\quad j = 0, 1, 2, \ldots, k-1.$

• The $k$**-diagonal semi-infinite Toda systems** are defined to be the Hamiltonian systems on $L^1_{S,k}$ associated to the Hamiltonians

$$I_l^{S,k}(\sigma) := I_l^S\left(\iota_{S,k}(\sigma)\right) = I_l\left((\iota_S \circ \iota_{S,k})(\sigma)\right)$$

for $\quad \sigma \in L^1_{S,k}$, where

$$I_l(\rho) := \frac{1}{l}\operatorname{Tr}\rho^l$$

for $\rho \in L_1$ and $l \in \mathbb{N}$.

# The semi-infinite Toda lattice

• For $k = 2$ we obtain semi-infinite Toda lattice which can be integrated by the theory of orthogonal polynomials.

• The phase-space $l^\infty \times l^1$

$$l^1 \ni \{p_n\}_{n=0}^{\infty} = p; \quad \|p\|_1 = \sum_{n=0}^{\infty} |p_n| < +\infty$$

$$l^\infty \ni \{q_n\}_{n=0}^{\infty} = q; \quad \|q\|_\infty = \sup_{n \in \mathbb{N} \cup \{0\}} |q_n| < +\infty$$

$$(l^1)^* = l^\infty$$

- The Poisson bracket:

$$f,\ g\ \in\ C^{\infty}(l^{\infty}\times l^{1})$$

$$\{f,g\}_{l^{\infty}\times l^{1}} := \sum_{n=0}^{\infty}\left(\frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} - \frac{\partial g}{\partial p_n}\frac{\partial f}{\partial q_n}\right)$$

$$\left(\frac{\partial f}{\partial p_n}\right)_{n=0}^{\infty} \in l^{\infty},\quad \left(\frac{\partial g}{\partial q_n}\right)_{n=0}^{\infty} \in (l^{\infty})^{*} \cong (l^{1})^{**}$$

$$\iota : l^{1}\ \hookrightarrow\ (l^{\infty})^{*}$$

The Poisson bracket $\{\cdot,\cdot\}_{l^{\infty}\times l^{1}}$ has sense if $\left(\frac{\partial f}{\partial p_n}\right), \left(\frac{\partial g}{\partial q_n}\right)\ \in\ l^{1}$.

- The Hamiltonian:

$$H_{Toda} = \frac{1}{2}\sum_{n=0}^{\infty} p_n^2 + \sum_{n=0}^{\infty}\nu_n e^{2(q_{n+1}-q_n)};\quad \{p_n\}_{n=0}^{\infty},\ \{\nu_n\}_{n=0}^{\infty} \in l^{1}$$

- The Hamilton equations:

$$\dot{q}_n = \{h, q_n\}_{l^\infty \times l^1} = p_n$$

$$\dot{p}_n = \{H_{Toda}, p_n\}_{l^\infty \times l^1} = \tag{0.15}$$
$$= -2\nu_{n-1} e^{2(q_n - q_{n-1})} + 2\nu_n e^{2(q_{n+1} - q_n)}$$

$$I_1(p, q) = \sum_{n=0}^{\infty} p_n \quad - \quad \text{integral of motion}$$

- The Banach Lie-Poisson space:

$$L^1 \ni \rho = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \\ 0 & b_1 & a_2 & \\ \vdots & & \ddots & \ddots \end{pmatrix}; \quad (a_n)_{n=0}^{\infty},\ (b_n)_{n=0}^{\infty} \in l^1$$

$$L^\infty \ni X = \begin{pmatrix} x_0 & y_0 & 0 & \cdots \\ 0 & x_1 & y_1 & \\ 0 & 0 & x_2 & \ddots \\ \vdots & & & \ddots \end{pmatrix}; \quad (x_n)_{n=0}^{\infty},\ (y_n)_{n=0}^{\infty} \in l^\infty$$

$$L^\infty \cong (L^1)^*; \qquad \langle X,\ \rho\rangle = Tr\ X\rho$$

$$F,\ G\ \in C^\infty(L^1)$$

$$DF(\rho),\ DG(\rho)\ \in\ (L^1)^* \cong L^\infty$$

- The Poisson bracket:

$$\{F, G\}_{L^1} := Tr(J[DF(\rho), DG(\rho)]) =$$

$$= \sum_{n=0}^{\infty} b_n \left[\frac{\partial F}{\partial b_n}\left(\frac{\partial G}{\partial a_{n+1}} - \frac{\partial G}{\partial a_n}\right) - \frac{\partial G}{\partial b_n}\left(\frac{\partial F}{\partial a_{n+1}} - \frac{\partial F}{\partial a_n}\right)\right]$$

- The momentum map:

$$J:\ l^\infty \times l^1 \ni (q,p) \mapsto \begin{pmatrix} p_0 & 0 & 0 & \cdots \\ \nu_0 e^{q_1-q_0} & p_1 & 0 & \\ 0 & \nu_1 e^{q_2-q_1} & p_2 & \\ \vdots & & \ddots & \ddots \end{pmatrix} \in L^1$$

is a Poisson map, i.e.

$$\{F,G\}_{L^1} \circ J = \{F\circ J, G\circ J\}_{l^\infty \times l^1}.$$

- Integrals of motion:

$$I_l(\rho) := \frac{1}{l} Tr(\rho + \rho^T - \rho_0)^l$$

where $\rho_0 = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & \\ 0 & 0 & a_2 & \\ \vdots & & & \ddots \end{pmatrix}$,

are in involution:

$$\{I_l, I_k\}_{L^1} = 0.$$

Thus we have

$$\{I_l \circ J, I_k \circ J\}_{l^\infty \times l^1} = 0$$

and

$I_1 \circ J(q,p) = \sum_{n=0}^{\infty} p_n$ - the total momentum

$I_2 \circ J(q,p) = H_{Toda}$ - the total energy

$I_k \circ J(q,p)$ - are integrals of motion for $H_{Toda}$ and $k > 2$.

• The Hamilton equations given by $I_l, \;\; l \in \mathbb{N}$ are

$$\frac{\partial J}{\partial t_l} = [J, B_l], \qquad (0.16)$$

where

$J := \rho + \rho^T - \rho_0$

$B_l := P_-(J^l) - (P_-(J^l))^T$

$P_-$ $-$projector on the lower triangular part of matrix.

Hamilton equations given by $I_l \circ J, \;\; l \in \mathbb{N}$ are

$$\begin{aligned} \tfrac{\partial q_n}{\partial t_l} &= \{I_l \circ J, q_n\}, \\ \tfrac{\partial p_n}{\partial t_l} &= \{I_l \circ J, p_n\}, \;\; n \in \mathbb{N} \cup \{0\} \end{aligned} \qquad (0.17)$$

The system (0.17) is obtained from (0.16) by reduction.

# Orthogonal polynomials and solutions of Toda hierarchy

• The operator $J$ given by

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \\ 0 & 0 & b_2 & a_3 & \\ \vdots & & & & \ddots \end{pmatrix}$$

has discrete simple spectrum $\lambda_0, \lambda_1, ... \in \mathbb{R}$, since $(a_n), (b_n) \in l^1$.

- Polynomials $P_n(\lambda)$ are orthonormal

$$\int P_n(\lambda)P_m(\lambda)d\sigma(\lambda) = \delta_{nm}$$

with respect to

$$d\sigma(\lambda) = \sum_{m=0}^{\infty} \mu_m \delta(\lambda - \lambda_m)d\lambda$$

satisfy the tree-term recurrence

$$\lambda P_n(\lambda) = b_{n-1}P_{n-1}(\lambda) + a_n P_n(\lambda) + b_n P_{n+1}(\lambda)$$

$$P_0(\lambda) = 1, \quad b_{-1} = 0.$$

- $\{|n\rangle\}_{n=0}^{\infty}$ - canonical basis

$$|\lambda_m\rangle := \sum_{n=0}^{\infty} P_n(\lambda_m)\,|n\rangle$$

$$J\,|\lambda_m\rangle = \lambda_m\,|\lambda_m\rangle$$

For projectors $\quad P_m := \frac{|\lambda_m\rangle\langle\lambda_m|}{\langle\lambda_m|\lambda_m\rangle}$ one has

$$(\frac{\partial}{\partial t_l}\lambda_n)P_nP_k - (\lambda_n - \lambda_k)\left[(\frac{\partial}{\partial t_l}P_n)P_k - P_nB_lP_k\right] = 0$$

where $n, k \in \mathbb{N} \cup \{0\},\ l \in \mathbb{N}$,
from the above

$$\begin{aligned} \tfrac{\partial}{\partial t_l}\lambda_m &= 0 \\ \tfrac{\partial}{\partial t_l}\mu_m &= 2\left[\lambda_m^l - \left(\textstyle\sum_{n=0}^{\infty}\mu_m\lambda_m^l\right)\right]\mu_m \end{aligned} \qquad (0.18)$$

- **Remark**: $t_l$ - the evolution of measure

$$d\sigma(\lambda) = \sum_{m=0}^{\infty} \mu_m \delta(\lambda - \lambda_m) d\lambda$$

is isospectral and

$$\mu_m = \langle \lambda_m | \lambda_m \rangle^{-1}.$$

The moments

$$\sigma_k := \int \lambda^k d\sigma(\lambda), \qquad \sigma_0 = 1$$

of $d\sigma(\lambda)$ satisfy

$$\frac{\partial}{\partial t_l} \sigma_k = 2(\sigma_{k+l} - \sigma_k \sigma_l), \qquad k, l \in \mathbb{N}. \quad (0.19)$$

It follows from (0.19) that exists $\tau = \tau(t_1, t_2, ...)$, such that

$$\sigma_k = \frac{1}{2}\frac{\partial}{\partial t_k}\log \tau$$

and

$$\frac{\partial^2 \tau}{\partial t_k \partial t_l} = 2\frac{\partial \tau}{\partial \tau_{k+l}}. \tag{0.20}$$

• The solution of (0.20) is given by:

$$\tau(t_1, t_2, ...) = \tau(0, 0, ...) \sum_{m=0}^{\infty} \mu_m(0, 0, ...) e^{2\sum_{l=1}^{\infty} t_l \lambda_m^l}$$

• The solution of (0.17) can be expressed by $\tau$-functions:

$$a_k = \frac{\beta_{k+1}}{\alpha_{k+1}} - \frac{\beta_k}{\alpha_k}$$
$$b_k = \frac{\sqrt{\alpha_{k-1}\alpha_{k+1}}}{\alpha_k}$$

where

$$\alpha_k = \det \begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{k+1} \\ & \cdots & \cdots & \\ \sigma_k & \sigma_{k+1} & \cdots & \sigma_{2k} \end{pmatrix}$$

$$\beta_k = \det \begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{k-1} & \sigma_{k+1} \\ \sigma_1 & \sigma_2 & \cdots & \sigma_k & \sigma_{k+2} \\ \cdots & \cdots & \cdots & & \\ \sigma_k & \sigma_{k+1} & \cdots & \sigma_{2k-1} & \sigma_{2k+1} \end{pmatrix}$$

$$\sigma_k = \frac{1}{2}\frac{\partial}{\partial t_k} \log \tau.$$

• The solution of Toda lattice equation is given by:

$$q_n(t) = q_n(0) + \frac{1}{2} \log \frac{\alpha_n(0)\beta_{n+1}(t)}{\beta_{n+1}(0)\alpha_n(t)}.$$