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Pseudo-fermionic coherent states.

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1. Introduction

- Until 1998 Hermiticity of the Hamiltonian was supposed to be the necessary condition for having real spectrum.
- 1998 C.M. Bender and S. Boettcher [1] have shown that with properly defined boundary conditions the spectrum of the non-Hermitian Hamiltonian :

$$H_\nu = p^2 + x^2 (ix)^\nu, (\nu \geq 0)$$

is real and positive

- As consequence, since this year the condition of the Hermiticity to have a real spectrum is relaxed and replaced by a more physical condition which is the PT-symmetry.
- 2002 A. Mostafazadeh [2] introduced the notion of pseudo-Hermiticity in order to establish the mathematical relation with the notion of PT-symmetry. He pointed out that all the PT-symmetric non-Hermitian Hamiltonians belonging to the class of pseudo-Hermitian Hamiltonians.

- By definition [2], a Hamiltonian H is called pseudo-Hermitian if it satisfies the relation:

$$H^{+} = \eta H \eta^{-1},$$

Where η is a linear Hermitian and invertible operator.

- One can also express this relation in the form: $H^{\#} = H$

  Where $H^{\#} = \eta^{-1} H^{+} \eta$ is the pseudo-adjoint of H .

- An interesting area where the pseudo-Hermiticity is illustrated is in the study of non-Hermitian two-level Hamiltonians (a two-level atom in interaction with an electromagnetic field with damping effects). The present work deals with this system.

- These simple Hamiltonian systems models accurately many physical systems in condensed matter, atomic physics, and quantum optics.

- Quantum optics provides a beautiful implementation of the coherent states formalism.

- Our goal is to extend the fermionic coherent states approach to two-level non-Hermitian Hamiltonians which are pseudo-Hermitian. The underlying number system is Grassmann algebra.

- Our system is described by the Following non-Hermitian Hamiltonian:

$$H = \frac{1}{2}\begin{pmatrix} -i\delta & \omega^* \\ \omega & i\delta \end{pmatrix}$$

- Where δ is a real constant which describes the damping effects.

- The complex quantity ω describes the radiation-atom interaction matrix element between the levels.

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Pseudo-Hermitian properties of H.

2. Pseudo-Hermitian properties of H :

- The trace of H is vanishing, and the determinant of H is real.

- Therefore H is pseudo-Hermitian according to the reference [3], “every 2×2 traceless matrix with real determinant is pseudo-Hermitian”.

- Indeed, the Hamiltonian H satisfies the pseudo-Hermiticity relation: $H^{+} = \eta H \eta^{-1}$, with η given explicitly by:

$$\eta = \begin{pmatrix} 1 & \frac{i\delta\omega^{*}}{|\omega|^{2}} \\ -\frac{i\delta\omega}{|\omega|^{2}} & 1 \end{pmatrix}$$

- The eigenvalues of H and the related complete biorthonormal system are given by:

$$E_1 = -\frac{\Omega}{2}, \qquad E_2 = \frac{\Omega}{2}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{-\omega^*\sqrt{\Omega+i\delta}}{|\omega|} \\ \sqrt{\Omega-i\delta} \end{pmatrix}, \quad |\psi_2\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\omega^*\sqrt{\Omega-i\delta}}{|\omega|} \\ \sqrt{\Omega+i\delta} \end{pmatrix}$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2\Omega^*}} \begin{pmatrix} \frac{-\omega^*\sqrt{\Omega^*-i\delta}}{|\omega|} \\ \sqrt{\Omega^*+i\delta} \end{pmatrix}, \quad |\phi_2\rangle = \frac{1}{\sqrt{2\Omega^*}} \begin{pmatrix} \frac{\omega^*\sqrt{\Omega^*+i\delta}}{|\omega|} \\ \sqrt{\Omega^*-i\delta} \end{pmatrix}$$

Where $\Omega = \sqrt{|\omega|^2 - \delta^2}$

- This complete biorthonormal system satisfies the following relations:

$$H|\psi_{1,2}\rangle = E_{1,2}|\psi_{1,2}\rangle, \qquad H^{+}|\phi_{1,2}\rangle = E^{*}_{1,2}|\phi_{1,2}\rangle$$

$$\langle\phi_1|\psi_1\rangle = \langle\phi_2|\psi_2\rangle = 1,$$

$$\langle\phi_1|\psi_2\rangle = \langle\phi_2|\psi_1\rangle = 0$$

$$|\phi_1\rangle\langle\psi_1| + |\phi_2\rangle\langle\psi_2| = 1,$$

$$|\psi_1\rangle\langle\phi_1| + |\psi_2\rangle\langle\phi_2| = 1.$$

- We point out here that we have two cases for the eigenvalues of *H*, namely:
- Case 1: real eigenvalues : $|\omega|^2 \geq \delta^2$ corresponding to the case where the dipole interaction is large compared to the damping effects. This case is very interesting in quantum optics [4].
- Case 2: pure imaginary eigenvalues : $|\omega|^2 < \delta^2$. The Hamiltonian *H* is still pseudo-Hermitian [4].

❖ In the present work we shall consider the case of the real eigenvalues (for physical reasons).

- After having diagonalized our pseudo-Hermitian Hamltonian H. We now embark on the construction of the pseudo-fermionic coherent states (PFCS) for H. The underlying number system is the Grassmann algebra.

3. Pseudo-fermionic coherent states.

Step 1:

Creation and annihilation operators for H.

3.1 Creation and annihilation operators for H.

- Now, let us introduce the annihilation operator b associated to the Hamiltonian H

$$b = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{-\omega^*(\Omega+i\delta)}{|\omega|} \\ \frac{\omega(\Omega-i\delta)}{|\omega|} & |\omega| \end{pmatrix}$$

- Its adjoint operator reads (Ω is real)

$$b^+ = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{\omega^*(\Omega+i\delta)}{|\omega|} \\ \frac{-\omega(\Omega-i\delta)}{|\omega|} & |\omega| \end{pmatrix}$$

- And its pseudo-Hermitian adjoint $b^\#$, is defined by

$$b^\# = \eta^{-1} b^+ \eta$$

- $b^{\#}$ takes the form

$$b^{\#} = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{\omega^{*}(\Omega - i\delta)}{|\omega|} \\ \frac{-\omega(\Omega + i\delta)}{|\omega|} & |\omega| \end{pmatrix}.$$

- $b^{\#}$ and b realize a pseudo-Hermitian generalization of the fermion algebra, namely,

$$b^2 = b^{\#2} = 0, \quad \{b, b^{\#}\} = bb^{\#} + b^{\#}b = 1$$

- One can verify that they raise and lower the eigenvalues of H by a quantity $\Omega = 2E$

- They act on the eigenstates $|\psi_i\rangle$ of H as follows:

$$b|\psi_1\rangle = 0, \qquad b|\psi_2\rangle = |\psi_1\rangle,$$

$$b^{\#}|\psi_2\rangle = 0, \qquad b^{\#}|\psi_1\rangle = |\psi_2\rangle,$$

- The operator b annihilates the lowest eigenstate $|\psi_1\rangle$, and $b^{\#}$ brings this state onto the upper eigenstate $|\psi_2\rangle$.

- Moreover, the Hamiltonian H is factorized in terms of the operators b and $b^\#$ to a form, similar to that of the free (boson) harmonic oscillator,

$$H = \Omega\left(b^\# b - \tfrac{1}{2}\right).$$

- Taking the Hermitian conjugate of both sides of this last expression of H we confirm the pseudo-Hermiticity of H (according to the definition $H^+ = \eta H \eta^{-1}$):

$$\begin{aligned} H^+ &= \Omega(b^+ \eta b \eta^{-1} - \frac{1}{2}) \\ &= \Omega \eta \eta^{-1}(b^+ \eta b \eta^{-1} - \frac{1}{2}) \eta \eta^{-1} \\ &= \eta H \eta^{-1}. \end{aligned}$$

❖ The above relations confirm that $b^{\#}$ and b are respectively the creation and annihilation operators of one degree of freedom of pseudo-Hermitian fermions [5].

• This result is confirmed in the Hermitian limit $\delta = 0$, $\eta = 1$ corresponding to a Hermitian Hamiltonian, as follow:

$$H^{+} = \eta H \eta^{-1}, \xrightarrow{\eta = 1} H^{+} = H$$

$$b^{\#} = \eta^{-1} b^{+} \eta \xrightarrow{\eta = 1} b^{\#} = b^{+}$$

❖ The pseudo-Hermitian generalization of the fermion algebra reduces to the usual fermion algebra.

- Having introduced the creation and annihilation operators, we now define the displacement operator.

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Step 2:

The displacement operator.

3.2 The displacement operator

- First, we define the displacement operator $D(\xi)$ for any set of complex Grassmannian variables ξ in the following way:

$$D(\xi) = \exp(b^{\#}\xi - \xi^{*}b)$$
$$= 1 + b^{\#}\xi - \xi^{*}b + \left(b^{\#}b - \frac{1}{2}\right)\xi^{*}\xi .$$

- The pseudo-Hermitian adjoint $D^{\#}$ is given by

$$D^{\#}(\xi) = \exp(\xi^{*}b - b^{\#}\xi)$$
$$= 1 + \xi^{*}b - b^{\#}\xi + \left(b^{\#}b - \frac{1}{2}\right)\xi^{*}\xi .$$

- These two operators satisfies the following displacement relations,

$$D^{\#}(\xi)\, b\, D(\xi) = b + \xi \mathbf{1} ,$$

$$D^{\#}(\xi)\, b^{\#} D(\xi) = b^{\#} + \xi^{*} \mathbf{1}$$

- Using the explicit formulas of D and $D^{\#}$, and the anticommutation relations between operators b, $b^{\#}$ and Grassmann variable ξ we establish that $D(\xi)$ are pseudo-unitary:

$$D^{\#}(\xi)D(\xi) = 1 = D(\xi)D^{\#}(\xi).$$

❖ Having introduced all the ingredients, we define now our coherent states.

Step 3:

Definition of pseudo-fermionic coherent states.

3.3 Definition of the pseudo-fermionic coherent states

- Now we define the pseudo-fermionic coherent states $|\xi\rangle$ as eigenstates of the annihilation operator b,

$$b|\xi\rangle = \xi|\xi\rangle .$$

The eigenvalue ξ is a complex Grassmannian variable.

- Similarly to the cases of Glauber bosonic coherent states [6] and of fermionic coherent states, our coherent states $|\xi\rangle$ can be constructed from the lowest (ground) eigenstate $|\psi_1\rangle$ of the Hamiltonian H, acting on it by the pseudo-unitary operator $D(\xi)$:

$$|\xi\rangle = D(\xi)|\psi_1\rangle$$

- By using the expression of the displacement operator $D(\xi)$, we may write the state $|\xi\rangle$ in the form:

$$\begin{aligned} |\xi\rangle &= D(\xi)|\psi_1\rangle \\ &= e^{(b^{\#}\xi - \xi^* b)}|\psi_1\rangle \\ &= e^{-\frac{1}{2}\xi^*\xi}\, e^{b^{\#}\xi}|\psi_1\rangle \\ &= e^{-\frac{1}{2}\xi^*\xi}(|\psi_1\rangle - \xi|\psi_2\rangle) . \end{aligned}$$

- The Hermitian adjoint of $|\xi\rangle$ is

$$\langle\xi| = e^{-\frac{1}{2}\xi^*\xi}(\langle\psi_1| + \xi^*\langle\psi_2|),$$

- By using the expression $D^\#(\xi)bD(\xi) = b + \xi\mathbf{1}$, we show that the coherent states $|\xi\rangle$ are eigenstates of the annihilation operator b,

$$\begin{aligned} b|\xi\rangle &= bD(\xi)|\psi_1\rangle \\ &= D(\xi)D^\#(\xi)bD(\xi)|\psi_1\rangle \\ &= D(\xi)(b+\xi)|\psi_1\rangle = D(\xi)\xi|\psi_1\rangle \\ &= \xi D(\xi)|\psi_1\rangle \\ &= \xi|\xi\rangle \end{aligned}$$

- And the inner product $\langle\xi|\xi\rangle$ is

$$\langle\xi|\xi\rangle = \langle\psi_1|\psi_1\rangle + (\langle\psi_2|\psi_2\rangle - \langle\psi_1|\psi_1\rangle)\xi^*\xi - 2i\mathrm{Im}(\xi\langle\psi_1|\psi_2\rangle) \neq 1.$$

❖ So that the states $|\xi\rangle$ are not normalized.

3.3.1 The Overcompleteness property

- For the Overcompleteness property, we have :

$$\int d\xi^* d\xi |\xi\rangle\langle\xi| = \int d\xi^* d\xi \left(|\psi_1\rangle\langle\psi_1| - \xi|\psi_2\rangle\langle\psi_1| + \xi^*|\psi_1\rangle\langle\psi_2| - \xi^*\xi(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)\right)$$

$$= (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) \neq \mathbf{1},$$

❖ So the Overcompleteness property of the coherent states $|\xi\rangle$ is not verified.

❖ c/c : The family of coherent states $|\xi\rangle$ constructed forms just one subset of the coherent states.

- The task is how to construct an overcomplete set of pseudo-fermionic coherent states for our system ?

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Step 4:

Construction of the second subset of coherent states.

3.4 Construction of the second subset of coherent states.

- The main idea to approach this problem is the use of the known transition from 'orthonormal system' of eigenstates of Hermitian Hamiltonian to the 'biorthonormal system' of states of pseudo-Hermitian Hamiltonians.

- With this idea in mind we introduce another continuous family of states namely the eigenstates $\widetilde{|\xi\rangle}$ of the operator $\tilde{b}$, that annihilates the dual state $|\phi_1\rangle$ of H^+,

$$\tilde{b}\widetilde{|\xi\rangle} = \xi\widetilde{|\xi\rangle},$$
$$\tilde{b}|\phi_1\rangle = 0, \quad \tilde{b}|\phi_2\rangle = |\phi_1\rangle.$$

- The operator $\tilde{b}$ (which is the annihilation operator of H^{+}) is given explicitly by,

$$\tilde{b} = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{-\omega^{*}(\Omega - i\delta)}{|\omega|} \\ \frac{\omega(\Omega + i\delta)}{|\omega|} & |\omega| \end{pmatrix}$$

- $\tilde{b}$ is related to the annihilation operator b of H by the relation

$$\tilde{b} = \eta b \eta^{-1}$$

- The creation operator $\tilde{b}^{\#\prime}$ of H^{+} is given explicitly by

$$\tilde{b}^{\#\prime} = \frac{1}{2\Omega}\begin{pmatrix} -|\omega| & \frac{\omega^{*}(\Omega + i\delta)}{|\omega|} \\ \frac{-\omega(\Omega - i\delta)}{|\omega|} & |\omega| \end{pmatrix}$$

❖ Indeed, the pair of pseudo-fermionic operators $\tilde{b}$ and $\tilde{b}^{\#\prime}$ realize also a pseudo-Hermitian generalization of the fermion algebra, namely,

$$\tilde{b}\,\tilde{b}^{\#\prime} + \tilde{b}^{\#\prime}\tilde{b} = 1,$$

$$\tilde{b}^{2} = (\tilde{b}^{\#\prime})^{2} = 0.$$

- Also, the Hamiltonian H^{+} is factorized in terms of the operators $\tilde{b}$ and $\tilde{b}^{\#\prime}$ in the usual form,

$$H^{+} = \Omega\left(\tilde{b}^{\#\prime}\tilde{b} - \tfrac{1}{2}\right).$$

❖ We follow a similar method which has been used before in the construction of the coherent states $|\xi\rangle$, to construct new subset of the coherent states $\widetilde{|\xi\rangle}$ associated to H^{+}.

- we introduce now the new displacement operators

$$\widetilde{D}(\xi) = e^{(\tilde{b}^{\#'}\xi - \xi^{*}\tilde{b})},$$

- Which satisfy the following displacement relation,

$$\widetilde{D}^{\#'}(\xi)\widetilde{D}(\xi) = \widetilde{D}(\xi)\widetilde{D}^{\#'}(\xi) = 1$$

$$\widetilde{D}^{\#'}(\xi)\,\tilde{b}\,\widetilde{D}(\xi) = \tilde{b} + \xi 1 .$$

- We construct now the second subset of coherent states $\widetilde{|\xi\rangle}$ according to the above described scheme, which are eigenstates of the new annihilation operator $\tilde{b}$

$$\begin{aligned}
\widetilde{|\xi\rangle} &= \widetilde{D}(\xi)|\phi_1\rangle, \\
&= e^{(\tilde{b}^{\#'}\xi - \xi^{*}\tilde{b})}|\phi_1\rangle, \\
&= e^{-\frac{1}{2}\xi^{*}\xi} e^{\tilde{b}^{\#'}\xi}|\phi_1\rangle \\
&= e^{-\frac{1}{2}\xi^{*}\xi}(|\phi_1\rangle - \xi|\phi_2\rangle) .
\end{aligned}$$

- The Hermitian adjoint of $\widetilde{|\xi\rangle}$ is

$$\widetilde{\langle\xi|} = e^{-\frac{1}{2}\xi^*\xi}(\langle\phi_1| + \xi^*\langle\phi_2|).$$

- By using the expression $\widetilde{D}^{\#'}(\xi)\tilde{b}\widetilde{D}(\xi) = \tilde{b} + \xi 1.$ we show that the coherent states $\widetilde{|\xi\rangle}$ are eigenstates of the annihilation operator $\tilde{b}$

$$\begin{aligned}
\tilde{b}\widetilde{|\xi\rangle} &= \tilde{b}\widetilde{D}(\xi)|\phi_1\rangle \\
&= \widetilde{D}(\xi)\widetilde{D}^{\#'}(\xi)\tilde{b}\widetilde{D}(\xi)|\phi_1\rangle \\
&= \widetilde{D}(\xi)(\tilde{b} + \xi)|\phi_1\rangle = \widetilde{D}(\xi)\xi|\phi_1\rangle \\
&= \xi\widetilde{D}(\xi)|\phi_1\rangle \\
&= \xi\widetilde{|\xi\rangle}.
\end{aligned}$$

- The scalar product between $\widetilde{\langle\xi|}\widetilde{|\xi\rangle}$ takes the form

$$\widetilde{\langle\xi|}\widetilde{|\xi\rangle} = \langle\phi_1|\phi_1\rangle + (\langle\phi_2|\phi_2\rangle - \langle\phi_1|\phi_1\rangle)\xi^*\xi - 2i\mathrm{Im}(\xi\langle\phi_1|\phi_2\rangle) \neq 1,$$

- while

$$\widetilde{\langle \xi|}\xi\rangle = \langle \phi_1|\psi_1\rangle + (\langle \phi_2|\psi_2\rangle - \langle \phi_1|\psi_1\rangle)\xi^*\xi - 2i\mathrm{Im}(\xi\langle \phi_1|\psi_2\rangle),$$
$$= \langle \phi_1|\psi_1\rangle = 1.$$

- And

$$\langle \xi\widetilde{|\xi\rangle} = \langle \psi_1|\phi_1\rangle + (\langle \psi_2|\phi_2\rangle - \langle \psi_1|\phi_1\rangle)\xi^*\xi - 2i\mathrm{Im}(\xi\langle \psi_1|\phi_2\rangle),$$
$$= \langle \psi_1|\phi_1\rangle = 1.$$

- This two last equations are obtained by using the biorthonormality of the system $\{|\psi_{1,2}\rangle, |\phi_{1,2}\rangle\}$ related to H which satisfies the relation:

$$\langle \phi_1|\psi_1\rangle = \langle \phi_2|\psi_2\rangle = 1,$$
$$\langle \phi_1|\psi_2\rangle = \langle \phi_2|\psi_1\rangle = 0.$$

$$|\phi_1\rangle\langle \psi_1| + |\phi_2\rangle\langle \psi_2| = 1,$$
$$|\psi_1\rangle\langle \phi_1| + |\psi_2\rangle\langle \phi_2| = 1.$$

- We said that $|\xi\rangle$ and $\widetilde{|\xi\rangle}$ are bi-normalized.
- And more generally,

$$\langle\xi_1\widetilde{|\xi_2\rangle} = \widetilde{\langle\xi_1|}\xi_2\rangle = \xi_1^*\xi_2 + \frac{1}{4}(2-\xi_1^*\xi_1)(2-\xi_2^*\xi_2),$$

- By means of the two type of states $|\xi\rangle$ and $\widetilde{|\xi\rangle}$ the resolution of the identity is realized now in the following way:

$$\int d\xi^* d\xi\, |\xi\rangle\widetilde{\langle\xi|} = \int d\xi^* d\xi\,(|\psi_1\rangle\langle\phi_1| - \xi|\psi_2\rangle\langle\phi_1| + \xi^*|\psi_1\rangle\langle\phi_2| - \xi^*\xi\mathbf{1}), = 1.$$

- And

$$\int d\xi^* d\xi\, \widetilde{|\xi\rangle}\langle\xi| = \int d\xi^* d\xi\,(|\phi_1\rangle\langle\psi_1| - \xi|\phi_2\rangle\langle\psi_1| + \xi^*|\phi_1\rangle\langle\psi_2| - \xi^*\xi\mathbf{1}) = 1.$$

- We said that $|\xi\rangle$ and $\widetilde{|\xi\rangle}$ satisfies the bi-overcompleteness property.

5. Conclusion

- We have obtained that the system of one-mode pseudo-fermionic coherent states consists of two subsets, namely $\{|\xi\rangle\}$ and $\{\widetilde{|\xi\rangle}\}$.

- This continuous system of pseudo-fermionic coherent states $\{|\xi\rangle, \widetilde{|\xi\rangle}\}$ forms a bi-normal and bi-overcomplete system.

- Similarly the two sets of pseudo-unitary operators $D(\xi), \widetilde{D}(\xi)$ are bi-unitary:

$$D(\xi)\widetilde{D}^{+}(\xi) = 1 = \widetilde{D}^{+}(\xi)D(\xi).$$

- We note finally that In the Hermitian limit of $\eta = 1 \Rightarrow H = \eta^{-1}H^{+}\eta = H^{+}$ our pseudo-fermionic coherent states and all related formulas recover standard fermionic coherent states obtained previously in references [7,8].

References

[1] C. M. Bender, S. Boettcher, Phys. Rev. Lett. **80**, 5243-5246 (1998).

[2] A. Mostafazadeh, J. Math. Phys. **43**, 205-214 (2002).

[3] A. Mostafazadeh, J. Math. Phys. **43**, 6343-6352 (2002).

[4] Y. Ben-Aryeh, A. Mann, I. Yaakov, J. Phys. A: Math. Gen. **37**, 12059-12066 (2004).

[5] A. Mostafazadeh, J. Phys. A: Math. Gen. **37**, 10193-10207 (2004).

[6] R. J. Glauber, Phys. Rev. Lett. **10**, 84-86 (1963)., R. J. Glauber, Phys. Rev. **130**, 2529-2539 (1963)., R. J. Glauber, Phys. Rev. **131**, 2766-2788 (1963).

[7] K. E. Cahill, R J. Glauber, Phys. Rev. A **59**, 1538-1555 (1999).

[8] C. J. Lee, Phys. Rev. A **46**, 6049-6051 (1992).