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Geometry of the Shilov boundary of bounded symmetric domains

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II.1 Bergman metrics

$\mathcal{D}$ a domain in $\mathbb{E}$

$$\mathcal{H}(\mathcal{D}) = \{f : \mathcal{D} \longrightarrow \mathbb{C}, f holomorphic, \int_{\mathcal{D}} |f(z)|^2 d\lambda(z) < \infty\}$$

For w in $\mathcal{D}$, consider $\mathcal{H}(\mathcal{D}) \ni f \longmapsto f(w)$.

This is a continuous linear form on $\mathcal{H}(\mathcal{D})$. Hence

$$f(w) = \int_{\mathcal{D}} f(z)\overline{K_w(z)} d\lambda(z) = \int_{\mathcal{D}} f(z)\overline{k(z,w)} d\lambda(z)$$

The kernel $k(z, w)$ is called the *Bergman kernel* of the domain $\mathcal{D}$. It satisfies :

$k(z, w)$ is holomorphic in z and conjugate holomorphic in w

$$k(w, z) = \overline{k(z, w)}$$

$$k(z, w) = J_\Phi(z)\ k(\Phi(z), \Phi(w))\ \overline{J_\Phi(w)}$$

for Φ a holomorphic diffeomorphism of $\mathcal{D}$ and $J_\Phi(.)$ is its Jacobian.

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Fact : for all z in $\mathcal{D}$, $k(z,z) > 0$ and the formula

$$h_z(\xi,\eta) = \partial_\xi \overline{\partial_\eta} \log k(u,w)_{u=z,w=z}$$

defines a Hermitian metric on $\mathcal{D}$ (the *Bergmann metric*). The metric is invariant under holomorphic diffeomorphisms of $\mathcal{D}$.

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II.2 Bounded symmetric domain

A bounded domain $\mathcal{D}$ is said to be *symmetric* ($\mathcal{D}$ is also called a Cartan domain) if, for every z in $\mathcal{D}$, there exists an involutive biholomorphic diffeomorphism s_z of $\mathcal{D}$ such that z is an isolated fixed point of s_z.

Use of Bergman metric implies : $\mathcal{D}$ is a Hermitian symmetric space, and $\mathcal{D} \simeq G/K$, where G is the neutral component of the group of holomorphic diffeo. of $\mathcal{D}$, and K the stabilizer of some fixed origin o.

$\mathcal{D}$ is said to be *circled* if 0 is in $\mathcal{D}$, and $\mathcal{D}$ is stable by $z \mapsto e^{i\theta}z$.

Theorem 1. *(JP Vigué) Any bounded symmetric space is holomorphically equivalent to a (bounded symetric) circled domain.*

Let $\mathcal{D}$ be a bounded circled symmetric domain. Choose 0 as origin in $\mathcal{D}$. Then the stabilizer K of 0 in G acts by linear transforms on $\mathbb{E}$, and preserves the inner product h_0. Hence K can be viewed as a closed subgroup of $\mathbb{U}(\mathbb{E}, h_0)$. The symmetry s_0 is given by $z \mapsto -z = e^{i\pi}z$ and belongs to K. The map $g \mapsto s_0 \circ g \circ s_0$ is a Cartan involution of G, with K as set of fixed points.

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$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition of $\mathfrak{g}$.

Any X in $\mathfrak{p}$ induces a holomorphic vector field ξ_X in $\mathcal{D}$, which can be regarded as a holomorphic map $\xi_X : \mathcal{D} \to \mathbb{E}$. The map $X \mapsto \xi_X(0)$ yields a (real) isomorphism of $\mathfrak{p}$ with $\mathbb{E}$, which is moreover K equivariant. The *bracket* of two holomorphic vectors fields ξ and η is the holomorphic vector field $[\xi, \eta]$ defined by

$$[\xi, \eta](z) = d\eta(z)\xi(z) - d\xi(z)\eta(z)$$

. For X, Y in $\mathfrak{g}$, one has the relation $\xi_{[X,Y]} = -[\xi_X, \xi_Y]$.

For u in $\mathbb{E}$, denote by ξ_u the unique holomorphic vector field induced by some element of $\mathfrak{p}$ such that $\xi_u(0) = u$.

Proposition 2. *Let v be in $\mathbb{E}$. Then, for z in $\mathcal{D}$,*

$$\xi_v(z) = v - Q(z)v$$

where $Q(z)$ is a $\mathbb{C}$-conjugate linear map of $\mathbb{E}$, and $z \mapsto Q(z)$ is a homogeneous quadratic polynomial of degree 2.

For u, v in $\mathbb{V}$, set $Q(u,v) = Q(u+v) - Q(u) - Q(v)$ (polarized symmetric form of Q, except for a factor $1/2$), and for x, y, z in $\mathbb{E}$, let

$$\{x, y, z\} = Q(x, z)y$$

.

Theorem 3. *The formula above defines on $\mathbb{E}$ a structure of positive Hermitian Jordan triple system (PHJTS) isomorphic to the Jordan sytem constructed on $\mathfrak{p}_+$.*

II.3 The spectral norm on $\mathbb{E}$

Let $\mathbb{E}$ be a PHJTS. For x, y in $\mathbb{E}$, let $L(x, y)$ be the $\mathbb{C}$-linear operator defined on $\mathbb{E}$ by $L(x, y)z = \{x, y, z\}$.

A real subspace W of $\mathbb{E}$ is said to be *flat* if

$$(1) \qquad \{W, W, W\} \subset W$$

$$(2) \qquad \text{for all x,y} \in W, \quad \{x, y, z\} = \{y, x, z\}$$

If W is flat, observe that (2) implies that the restriction of $\Re\tau(x, y)$ to W is a Euclidean inner product on W, and, for x, y in W, the restriction $\widetilde{L(x,y)}$ of $L(x,y)$ to W is a symmetric operator for this inner product. Moreover (2) implies that these restrictions mutually commute one to each other. Hence they have a common diagonalization.

An element c of $\mathbb{E}$ is said to be a *tripotent* if it satisfies

$$\{c,c,c\} = 2c$$

Two tripotents c and d are said to be *orthogonal* if $L(c,d) = 0$. If this is the case, then $c+d$ is a tripotent.

If c is a tripotent, then $L(c,c)$ is seladjoint, and its eigenvalues belong to $\{2,1,0\}$, so that there is a corresponding decomposition of $\mathbb{E}$ as $\mathbb{E} = \mathbb{E}_2 \oplus \mathbb{E}_1 \oplus \mathbb{E}_0$ (Peirce decomposition w.r.t. c).

Theorem 4. *Let $c_1, c_2, \dots, c_s$ a family of mutually orthogonal tripotents. Then $W = \mathbb{R}c_1 \oplus \mathbb{R}C_2 \oplus \cdots \oplus \mathbb{R}c_s$ is a flat subspace of W. Conversely, let W be a flat subspace. Then there exists a family $c_1, c_2, \dots, c_s$ of mutually orthogonal tripotents such that $W = \mathbb{R}c_1 \oplus \mathbb{R}c_2 \oplus \cdots \oplus \mathbb{R}c_s$. Moreover the family is unique up to order and sign.*

If x is any element in $\mathbb{E}$, its *odd powers* are defined by induction : $x^{(2p+1)} = Q(x)x^{(2p-1)}$. The real vector space $\mathbb{R}[x]$ generated by the odd powers of x form a flat subspace, and hence, by the previous result, there exists a unique family c_1, c_2, c_s of mutually orthogonal tripotents, and real positive numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s$ such that $x = \lambda_1 c_1 + \lambda_2 c_2 \cdots + \lambda_s c_s$. The λ_j's are called the *eigenvalues* of x. The *spectral norm* of x is by definition the tlargest eigenvalue of x, denoted by $|x|$. It can be shown that $x \mapsto |x|$ is actually a (complex Banch) norm on $\mathbb{E}$.

Theorem 5. *Let $\mathcal{D}$ be a bounded circled symmetric domain in $\mathbb{E}$. Let $\{.,.,.\}$ be the induced structure of PHJTS on $\mathbb{E}$, and let $|.|$ the corresponding spectral norm on $\mathbb{V}$. Then $\mathbb{D} = \{x \in \mathbb{E}, |x| < 1$. Conversely, let $\mathbb{E}$ be a PHJTS. The open unit ball for the spectral norm is a bounded symmetric domain.*