III The Shilov boundary

V Positive Hermitian Jordan Triple System

$\{x,y,z\}$ C-linear in x and z, conjugate linear in y
symmetric in (x,z)
a quintic identity

$\tau(x,y) = \mathrm{tr}\,(z \to \{x,y,z\})$ positive definite hermitian form

c tripotent $\{c,c,c\} = 2c$

$|x|$ spectral norm of x

D unit ball for the spectral norm

$G = \mathrm{Hol}(D)^0$, K stabilizer of 0 in G.

Example. $V = \mathrm{Mat}(p,q,\mathbb{C})$ $(p \leq q,\ n=p+q)$

$\{x,y,z\} = xy^*z+zy^*x$, $|x| = \|x\|_{op}$

$D = \{x \text{ in } V,\ |x|<1\}$, $G = PU(p,q)$, $K = P(U(p)xU(q))$

$k(z,w)= \det(1_p-zw^*)^{-n}$

III.1 Tripotents and Peirce frames

Let V be a PHJTS, with triple product {.,.,.}. Assume, for convenience, that V is simple (i.e. cannot be written as a sum of two PHJTS).

Recall that a tripotent is an element c which satisfies {c,c,c} = 2c.

There is a (partial) order on tripotents : if c and d are two tripotents, then say $c < d$ if there exists a tripotent $f \neq 0$, orthogonal to c and such that $d = c + f$.

A tripotent c is *primitive* (or minimal) if c can not be written as a sum of two (non zero) tripotents. Any tripotent can be written as a sum of primitive orthogonal tripotents.

Any two minimal tripotents are conjugate under an automorphism of V.

A *Peirce frame* is a maximal set of orthognal primitive tripotents. Any two Peirce frames are conjugate under an automorphism of V. In particular, the number of elements is the same for all frames (call it the *rank* of V).

Let c be a tripotent. Then TFAE

(i) $c = c_1 + \ldots + c_r$, where $(c_1, \ldots, c_r)$ is a Peirce frame

(ii) c is a maximal tripotent

(iii) $V = V_2(c) + V_1(c)$ (i.e. $V_0(c) = 0$)

(iv) $R c_1 + R c_2 + \ldots + R c_r$ is a maximal flat space in V.

III.2 The Shilov boundary

Let D be a domain in some complex vector space.

The *Shilov boundary* S of D is the smallest closed subset of the boundary of D, for which the maximum principle for the modulus of holomorphic function applies. The Shilov boundary may be much smaller than the topological boundary.

Example 1. Let D be the product of two copies of the complex unit disc. Then the Shilov boundary of D is the product of two copies of the unit circle, as can be seen by applying *twice* the maximum principle w.r.t. each variable.

Example 2. Let D be the unit ball in Mat(p,q) with $p \leq q$, then

- x is in the topological boundary of D iff 1 is an eigenvalue of xx^*
  - x is in the Shilov boundary iff $xx^* = Id_p$

Example 3. The Siegel disc and the Lagrangian manifold

Let $V = \mathrm{Symm}(r, \mathbb{C})$ be the PHJTS, with product

$$\{x,y,z\} = xy^*z+zy^*x \; .$$

$$D = \{x \text{ in } V \; ; \; 1-xx^* >> 0\}$$

The group G is $Sp(2r, \mathbb{R})$(mod$\{\pm 1\}$), and K is isomorphic to U(r) acting on V by $(u,X) \to uXu^t$.

D is called the *Siegel disc*. Its Shilov boundary is

$$S = \{ \sigma \text{ in } V \; ; \; \sigma \sigma^* = 1\}.$$

S is isomorphic to the Lagrangian manifold (also to U(r)/O(r)).

Recall. (E, ω) a real symplectic vector space , of dimension 2r.
A *Lagrangian* L is a maximal totally isotropic vector subspace of E (hence of dimension r). The *Lagrangian manifold* is the set of all lagrangians. It sits in the Grassmanian of r-subspaces in E.

The Shilov boundary S of the open unit ball D in V can be described in the following equivalent ways :

i) S is the set of maximal tripotents

ii) S is the set of extremal points of the closed open ball (as a convex set)

iii) S is the set of points in the closed unit ball which are at maximal distance of the origin for the distance associated to Hermitian form τ.

III.3 Action of G on S and S x S

The action of a holomorphic diffeomorphism of D always extend to some neigh'd of the closure of D. Hence the action of G extends to the closure of D. In particular, G acts on S.

Proposition 1 S is a connected compact manifold. G acts transitively on S, and S is the unique closed G-orbit in the boundary of D. K (a maximal compact subgroup of G) acts already transitively on S. The stabilizer of a point in S is a (maximal) parabolic subgroup of G.

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Proposition 2 G has a (unique) open orbit in S x S.

A pair (σ,τ) in S x S in the open orbit is said to be *transverse*▷

Example Let S be the Lagrangian manifold. A pair of Lagrangians (L_1, L_2) is tranverse iff $L_1 \cap L_2 = \{0\}$. The symplectic group is transitive on pairs of transverse Lagrangians (Darboux).

A pair (σ,τ) in $S \times S$ is transverse

- iff there exists a geodesic $\gamma(t)$ in D such that

$$\gamma(+\infty) = \sigma, \ \gamma(-\infty) = \tau.$$

- iff the Bergman kernel extends by continuity to (σ,τ) (i.e. $k(\sigma,\tau)$ is defined).

III.4 Euclidean Jordan algebra

A Euclidean Jordan algebra is a Euclidean vector space $(W, \langle\ ,\ \rangle)$ with a bilinear product $x.y$ and a unit e such that

i) $x.y = y.x$

ii) $x^2(x.y) = x.(x^2.y)$ (weak associativity)

iii) $e.x = x.e = x$

iv) $\langle x.y,z\rangle = \langle x,y.z\rangle$

Example $W = \mathrm{Symm}(r,\mathbb{R})$, $x.y = 1/2(xy+yx)$, $e = \mathrm{id}$, $\langle x.y\rangle = \mathrm{tr}(xy)$.

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Example W = Symm(r,R), x.y = 1/2(xy+yx), e = id, $\langle x.y\rangle$ =t r(xy).

Hermitification of a Euclidean Jordan algebra W.
Let W be the complexification of W. Extend the Jordan product in a C-linear way. On W define

$$\{x , y , z\} = x . (\bar{y} . z) + z . (\bar{y} . x) - \bar{y} . (x . z) .$$

Then W is a PHJTS (called the Hermitification of W).

Example. W = Symm(r, C), {x,y,z} = (xy*z+zy*x).

III.5 Tube-type domains

D is said to be of *tube type* if S is totally real (equivalently if $\dim_R S = \dim_C V$).

Propostion 1. Let V be a PHJTS, and let D be its unit ball for the spectral norm. Then D is of tube type if and only if V is the hermitification of some Euclidean Jordan algebra.

Let W be a Euclidean Jordan algebra, W its hermitification. Then D is holomorphically equivalent to the tube V+iΩ through the *Cayley transform*, where Ω is the (interior of) the cone of squares $\{x^2$, x in V$\}$. The Cayley transform extends to a dense open of S and maps it to V+i0.

Example 1. V= Symm(r,R)

The cone Ω is the set of positive definite matrices.

The Siegel disc {z in Symm(r,C), 1-zz*>>0} is of tube type, holomorphically equivalent to the *Siegel upper half plane*

{x+iy, x, y in Symm(r,R), y >> 0}

through the Cayley transform $c(z) = i\,(1+z)\,(1-z)^{-1}$.

The Shilov boundary is S = {z in Symm(r,C), $z^* = z^{-1}$}. S is isomorphic to the Lagrangian manifold.

Example 2 The matrix unit ball $I_{p,q}$ is of tube-type iff p=q.

The associated PHJTS is Mat (p,q, ℂ). If p=q, then the PHJTS is the Hermitification of the Euclidean Jordan algebra

Herm(p,ℂ) with x.y = 1/2 (xy+yx) and <x,y>= Re(tr xy)

The Shilov boundary is S = U(p).

Proposition 2 If D is of tube-type, then S is a compact Riemannian symmetric space K/L.

III.4 Action of G on S x S x S

Proposition 4 Let D be a bounded symmetric domain of tube type. The action of G on S x S x S has a finite number of orbits, and in particular (r+1) open orbits.

Let V be the Euclidean Jordan algebra,to which D is associated. Let $(c_1, c_2, \ldots, c_r)$ be a Peirce frame such that $e = c_1 + c_2, \ldots + c_r$. For $0 \leq k \leq r$, let $e_k = c_1 + \ldots + c_k - c_{k+1} - \ldots - c_r$. Then

$$(e, -e, ie_k) \ (0 \leq k \leq r)$$

is a set of representatives of the open orbits.

N.B. If D is not of tube-type, S is not a symmetric space of U, and there are infinitely many G-orbits and no open G-orbit in S x S x S .

In the tube-type case, S is a Riem. Symmetric space. A maximal flat torus of S is of the form

$$T = \{ \xi_1 c_1 + \xi_2 c_2 + \ldots + \xi_r c_r,\ |\xi_j|=1,\ 1 \leq j \leq r \}$$

where $(c_1, c_2, \ldots, c_r)$ is Peirce frame.

Proposition 5 (KH Neeb, JLC '07) Let D be a bounded symmetric domain of tube type. Fix a maximal torus T in S. Let $\sigma_1, \sigma_2, \sigma_3$ be in S.

Then there exists g in G such that $g(\sigma_1), g(\sigma_2), g(\sigma_3)$ belong to T.

Example. Normal form of a triplet of Lagrangians .

Let $L^{(1)}, L^{(2)}, L^{(3)}$ be three arbitray Lagrangians in E. Then there exists a symplectic basis (e_j,f_j) of E such that

$$L^{(k)} = \text{span}\ \{ \cos \theta_1^{(k)} e_1 + \sin \theta_1^{(k)} f_1, \cos \theta_2^{(k)} e_2 + \sin \theta_2^{(k)} f_2, \ldots, \cos \theta_r^{(k)} e_r + \sin {}_r^{(k)} f_r \}$$

for k=1,2,3.

List of simple Euclidean Jordan algebras, bounded domains of tube type and their Shilov boundaries

V	D	S
Symm(r,$\mathbb{R}$)	unit ball in Symm(r,$\mathbb{C}$)	Lagrangian manifold
Herm(r,$\mathbb{C}$)	unit ball in Mat(r, $\mathbb{C}$)	U(r)
Herm(r,$\mathbb{H}$)	unit ball in Skew(2r,$\mathbb{C}$)	U(2r)/SU(m,$\mathbb{H}$)
$\mathbb{R} \times \mathbb{R}^{d-1}$ (*)	Lie ball in $\mathbb{C}^d$	$(U(1) \times S^{d-1}) / Z_2$
Herm(r,$\mathbb{O}$)	$E_{7(-25)}/U(1)E_6$	$U(1)E_6 / F_4$

(*) $(\lambda, x) . (\mu, y) = (\lambda + \mu + \langle x,y \rangle, \mu x + \lambda y)$