IV An invariant for triples and the Maslov index

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IV.1 The symplectic area

Let V be a PHJTS, and D the associated bounded symmetric domain, $G = \mathrm{Hol}(D)^0$. Recall D has a G-invariant metric (the Bergman metric). The associated *Kähler form* ω is the differential form of degree 2 defined by

$$\omega_z(\xi , \eta) = g_z(\xi , J_z \eta) , \quad z \text{ in } D, \ \xi , \eta \text{ in } V .$$

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Let z_1, z_2 be two points in D. There is a unique geodesic arc from z_1 to z_2 (a consequence of the negative curvature).
Let z_1, z_2, z_3 be three points in D. Form the *geodesic triangle* $T(z_1, z_2, z_3)$. Then let

$$A(z_1, z_2, z_3) = \int_\Sigma \omega$$

where Σ is any surface with $\partial\Sigma = T(z_1, z_2, z_3)$.
The integral does not depend on Σ and is called the *symplectic area* of the triangle.

The symplectic area has the following properties

i) $A(g(z_1), g(z_2), g(z_3)) = A(z_1, z_2, z_3)$, for g in G

ii) $A(z_{\tau(1)}, z_{\tau(2)}, z_{\tau(3)}) = \mathrm{sign}(\tau)\, A(z_1,z_2,z_3)$

for τ any permutation of 1,2,3

iii) (cocycle property) for all z_1,z_2,z_3, z_4 in D,

$$A(z_1,z_2,z_3) = A(z_1,z_2,z_4) + A(z_2,z_3,z_4) + A(z_3,z_1,z_4)$$

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Theorem (Domic-Toledo ‘87, Oersted-JLC ‘03)

$$A(z_1, z_2, z_3) = -(\arg k(z_1,z_2) + \arg k(z_2,z_3) + \arg k(z_3,z_1))$$

where $k(z,w) = k_D(z,w)^{2/p}$ (k_D the Bergman kernel), and p is an explicit integer (called the genus of D).

Observations : D is simply connected, $k(z,w) \neq 0$ on DxD and $k(z,z) > 0$ for z in D, so that there is a unique continuous determination of arg k(z,w) which takes value 0 on the diagonal.

For the unit disc in $\mathbb{C}$ with the Poincaré-Bergman metric, this is equivalent to the formula for the area of a geodesic triangle

$$A(a, b, c) = \pi - (\alpha+\beta+\gamma),$$

where α,β,γ are the angles at the summits a,b,c of the triangle.

Consequence of the formula :

$$-r\,\pi < A(z_1, z_2, z_3) < r\,\pi$$

for all z_1,z_2,z_3 in D, where r is the rank of the symmetric space.

IV-2 Passing to the limit

Theorem (Oersted-JLC ‘03, JLC ‘07)

For any σ_1 , σ_2 , σ_3 in S x S x S, let

$$i\,(\sigma_1 , \sigma_2 , \sigma_3) = 1/\pi \ \lim A\,(z_1, z_2, z_3)$$

as z_j tends to σ_j (j=1,2,3).
The limit exists with no restriction on the way z_j approaches σ_j if the σ_j are mutually tranverse. If not, some restriction is needed (“ kind of “ radial approach).

Theorem The function i satisfies the following properties

i) $i\,(g(\sigma_1), g(\sigma_2), g(\sigma_3)) = i(\sigma_1, \sigma_2, \sigma_3)$ for any g in G

ii) i is skew-symmetric w.r.t. permutations of the σ_j 's.

iii) For all $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ in S,

$$i\,(\sigma_1, \sigma_2, \sigma_3) = i\,(\sigma_1, \sigma_2, \sigma_3) + i\,(\sigma_2, \sigma_3, \sigma_4) + i\,(\sigma_3, \sigma_1, \sigma_4)$$

(cocycle relation)

iv) $-r \leq i\,(\sigma_1, \sigma_2, \sigma_3) \leq r$

and these inequalities are optimal (bounds attained).

IV-3 Elie Cartan's invariant : an example for the non-tube type case.

In 1932 E. Cartan constructed an invariant for triples in the unit sphere S in $\mathbb{C}^2$ under the action of the ''complex conformal'' group PU(2,1). Another realization of S is needed.

Consider $\mathbb{C}^3$ with the Hermitian form

$$h((z,x,y),(z,x,y)) = |z|^2 - |x|^2 - |y|^2$$

Then the map $(x,y) \to \mathbb{C}\,(1,x,y)$ is a 1-to-1 correspondance of S with the space of complex isotropic lines in $\mathbb{C}^3$. Then G = PU(h) = PU(2,1,$\mathbb{C}$) acts naturally on this space, yielding an action of G on S. Consider three distinct isotropic lines L_1, L_2, L_3 and for v_1 in L_1, v_2 in L_2, v_3 in L_3 let

$$J(v_1, v_2, v_3) = h(v_1, v_2)\, h(v_2, v_3)\, h(v_3, v_1)\,.$$

Now $J(\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3) = |\lambda_1|^2 |\lambda_2|^2 |\lambda_3|^2 J(v_1, v_2, v_3)$, hence $\arg J(v_1, v_2, v_3)$ defines a function on SxSxS, which is, by construction, invariant under G. This invariant coincides with ours (up to normalization).

IV-4 The Maslov triple index : an example for the tube-type case.

The geometry of G-orbits in SxSxS depends highly on the type of the bounded symmetric domain (tube-type vs non-tube type).

Proposition. Let D be a bounded irreducible symmetric domain *of tube type*. The action of G on S x S x S has a finite number of orbits, and in particular (r+1) open orbits, where r is the rank of D.

If D is not of tube type, then there are infinitely many G-orbits in S x S x S and none is open .

Fact. Assume D is of tube type. Let e be an origin in S, and let L be the stabilizer of e in K. Then S is isomorphic to K/L and S is a compact Riemannian symmetric space.

If D is not of tube type, then S is not a Riemannian symmetric space.

Proposition (KH Neeb, JLC '07) Let D be a bounded symmetric domain of tube type. Fix a maximal torus T in S. Let σ_1, σ_2, σ_3 be in S. Then there exists g in G such that

$$g(\sigma_1), g(\sigma_2), g(\sigma_3) \text{ belong to } T .$$

Example. Normal form of a triplet of Lagrangians .

Let $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ be three arbitrary Lagrangians in E. Then there exists a symplectic basis (e_j,f_j) (i.e. $\omega(e_j,f_j) = -\omega(e_j,f_j) = 1$ and all the others $\omega(e_i,f_j)$ are 0) such that for k=1,2,3, $L^{(k)}$ has basis of the form

$$\cos \theta_1^{(k)} e_1+\sin \theta_1^{(k)} f_1, \cos \theta_2^{(k)} e_2+ \sin \theta_2^{(k)} f_2, \ldots, \cos \theta_r^{(k)}e_r+ \sin {}_r^{(k)}f_r$$

Theorem. Let D be ot tube-type. Then the $i(\sigma_1 , \sigma_2 , \sigma_3)$ takes only the values -r, -r+1,…, r-1, r . If D is not of tube-type, then $i(\sigma_1 , \sigma_2 , \sigma_3)$ takes all values in [-r , r].

Example. The Maslov index

Kashiwara's definition of the Maslov index :

Let L_1, L_2, L_3 three Lagrangians in a real symplectic vector space (E,ω). Consider on the (abstract) product $L_1 x\ L_2 x\ L_3$ the quadratic form Q defined by :

$$Q((v_1 , v_2 , v_3)) = \omega (v_1 , v_2) + \omega (v_2 , v_3) + \omega (v_3 , v_1)$$

Set $\quad j (L_1, L_2, L_3)$ = signature of Q .

This defines a Sp(E)-invariant Z-valued function on S x S x S, where S is the Lagrangian manifold.

Theorem. Let S be the Shilov boundary of the Siegel unit disc, isomorphic to the Lagrangian manifold. Then our invariant coincides with the Maslov index.

IV- 5 The Maslov index for paths.

Originally, Maslov introduced a notion now called *Maslov's index for paths*. It associates to a path in the Lagrangian manifold (i.e. a 1- parameter family of Lagrangians) an integer (there are some restrictions on the endpoints depending on the choice of an origin in S). Maslov's work was continued by Arnold, and later by Leray. This approach is based on algebraic topology. In particular, the index is invariant by homotopy. A slightly different approach (but closely related) was introduced by Souriau.

The space S is not simply connected. In fact, $\pi_1(S) = Z$. Let Σ be the universal covering of S. Then Souriau constructs a Z - valued function m on $\Sigma \times \Sigma$, which is invariant, skew symmetric, and such that, for any three points σ_1 , σ_2 , σ_3 in Σ the sum

$$m(\sigma_1 , \sigma_2) + m(\sigma_2 , \sigma_3) + m(\sigma_3 , \sigma_1)$$

depends only on the projections of σ_1, σ_2 and σ_3 on S and is equal to the corresponding Maslov index of the three projections.

From this one could deduce the cocycle relation for the Maslov triple index.

Cohomological interpretation : the Maslov triple index is not cohomologically trivial on S x S x S. Lifted to Σ x Σ x Σ, it becomes trivial and can be written as the coboundary of m.

All these points of view (Maslov-Arnold-Leray and Souriau) can be generalized in the realm of Shilov boundaries of bounded symmetric domains of tube-type (Koufany-JLC ‘07).

IV- 6 List of simple Euclidean Jordan algebras, bounded domains of tube type and their Shilov boundaries

V	D	S
Symm(r,R)	unit ball in Symm(r,C)	Lagrangian manifold
Herm(r,C)	unit ball in Mat(r, C)	U(r)
Herm(r,H)	unit ball in Skew(2r,C)	U(2r)/SU(m,H)
$R \times R^{d-1}$ (*)	Lie ball in C^d	$(U(1) \times S^{d-1}) / Z_2$
Herm(r,O)	$E_{7(-25)}/U(1)E_6$	$U(1)E_6 / F_4$

(*) $(\lambda, x) . (\mu,y) = (\lambda\mu + \langle x,y\rangle, \mu x + \lambda y)$

IV- 7 Maslov index for beginners

See blackboard