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# Group Analysis of a Dynamical System Associated with Generalized Elastica


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# Aim

The aim of the present note is determination of analytic expressions for the embedding of the plane curves with curvatures $\kappa = f(x,y)$ of form

$$f(x,y) = ux^2 + 2wxy + vy^2 + kx + my + n$$

where $u$, $v$, $w$, $k$, $m$ and $n$ are real numbers and $(x,y)$ are Cartesian coordinates in the plane. This task is a specialization of the general case given in the lecture *Integrable Dynamical Systems Associated with Plane Curves* by the same authors that was presented by Dr. Vassilev yesterday.

# Approach

It is well known that each such curve is associated with a dynamical system of form

$$\ddot{x} + f(x,y)\dot{y} = 0, \quad \ddot{y} - f(x,y)\dot{x} = 0$$

describing motions of a particle of unit mass. Hence, the determination of analytic expressions for the embedding of the foregoing curves reduces to the problem of integrability of this system.

On the other hand, the integrability of a system of diferential equations is associated with the existense of variational symmetries (conservation laws) of this system.

An obvoious conservation law of the system reads

$$\dot{x}^2 + \dot{y}^2 = const.$$

Therefore, our aim reduces to determination of an additional conservation law.

# Recall from the preceding lecture

The Lagrangian, associated with this system is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \left[xy\left(ux + \frac{1}{2}wy + k\right) + \frac{1}{2}ny\right]\dot{x}$$
$$- \left[xy\left(vy + \frac{1}{2}wx + m\right) + \frac{1}{2}nx\right]\dot{y}$$

The condition for existence of additional conservation law, derived in the preceding lecture is

$$(ay + b)\frac{\partial}{\partial x}f(x, y) - (ax + c)\frac{\partial}{\partial y}f(x, y) = 0$$

where $a$, $b$ and $c$ are arbitrary real numbers.

# Two kinds of integrable systems

Let us recall here that given three real numbers $a$, $b$ and $c$, two sets of coefficients such that the corresponding dynamical system admits an additional conservation law are identified in the lecture in question, namely

$a = 0$ (Euler elastica)

$$u = v = w = 0, \quad k = c, \quad m = b$$
$$\kappa = f(x, y) = cx + by + n$$

$a \neq 0$ (Levy's elastica)

$$u = v = q, \quad w = 0, \quad k = \frac{2c}{a} q, \quad m = \frac{2b}{a} q$$
$$\kappa = f(x, y) = q\left(x^2 + y^2\right) + \frac{2q}{a}(cx + by) + n.$$

# Conservation laws

Omitting the details, the curvature and the corresponding two conservation laws for the case of Levy's elastica can be written in the form

$$\kappa = \frac{\sigma^2\left(x^2+y^2\right)-\lambda}{4\sigma}$$

$$\begin{aligned} \dot{x}^2+\dot{y}^2 &= 1 \\ (y\dot{x}-x\dot{y})+\left(\frac{\sigma^2\left(x^2+y^2\right)-2\lambda}{16\sigma}\right)\left(x^2+y^2\right) &= C \end{aligned}$$

where $\sigma$, $\lambda$ and $C$ are real constants ($\sigma \neq 0$).

Therefore, the embedding of the curves of curvatire can be obtained in analytic form.

# Determination of analytic expressions

Change of the dependent variables

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In terms of the new dependent variables the two conservation laws read

$$\begin{aligned} r^2\dot{\theta}^2 + \dot{r}^2 &= 1 \\ \dot{\theta} - \frac{\sigma^2 r^2 - 2\lambda}{16\sigma} + \frac{1}{r^2}C &= 0 \end{aligned}$$

These constitute a system of two equations for determination of the functions $\theta(s)$ and $r\,(s)$.

# Determination of the analytic expressions

Substituting $\dot{\theta}$ from the second equation into the first one yields

$$r^2\dot{r}^2 = -\frac{1}{256}\sigma^2 r^8 + \frac{1}{64}\lambda r^6 + \left(\frac{1}{8}\sigma C - \frac{1}{64}\frac{\lambda^2}{\sigma^2}\right) r^4 + \left(1 - \frac{1}{4}C\frac{\lambda}{\sigma}\right) r^2 - C^2.$$

On the other hand let us note that in the new variables the expression for the curvature is

$$\kappa = \frac{\sigma^2 r^2 - \lambda}{4\sigma}$$

implying that

$$\dot{\kappa} = \frac{\sigma}{2} r\dot{r}.$$

# Determination of analytic expressions

Solving the last two relations for $r$ and $\dot{r}$ and substituting the result in the first equation on the previous slide gives

$$\dot{\kappa}^2 = 2E - \frac{1}{4}\kappa^4 + \frac{1}{2}\mu\kappa^2 + \sigma\kappa$$

where

$$\begin{aligned} \mu &= \frac{1}{16\sigma^2}\left(\lambda^2 + 16\sigma^3 C\right) \\ E &= \frac{1}{2048\sigma^4}\left(+256\lambda\sigma^4 - \lambda^4 - 32\lambda^2\sigma^3 C - 256C^2\sigma^6\right). \end{aligned}$$

The foregoing equation is a first integral of the differential equation

$$2\ddot{\kappa} + \kappa^3 - \mu\kappa - \sigma = 0.$$

## Explicit expressions for the solutions of the obtained equation

Case I: two real ($\alpha < \beta$) and two complex ($\gamma = \bar{\delta} = -\frac{\alpha+\beta}{2} + i\eta$) roots of the polynomial:

Periodic solution

$$\kappa_1(s) = \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\,\mathrm{cn}(us, k)}{(A + B) - (A - B)\,\mathrm{cn}(us, k)}$$

Aperiodic solution

$$\kappa_2(s) = \zeta - \frac{4\zeta}{1 + \zeta^2 s^2}$$

## Explicit expressions for the solutions of the obtained equation

Case II: four real roots ($\alpha < \beta < \gamma < \delta$) of the polynomial:

$$\kappa_3(s) = \delta - \frac{(\delta - \alpha)(\delta - \beta)}{(\delta - \beta) + (\beta - \alpha)\,\mathrm{sn}^2(us, k)}$$

$$\kappa_4(s) = \beta + \frac{(\gamma - \beta)(\delta - \beta)}{(\delta - \beta) - (\delta - \gamma)\,\mathrm{sn}^2(us, k)}$$

where

$$u = \frac{1}{4}\sqrt{(\gamma - \alpha)(\delta - \beta)}, \qquad k = \sqrt{\frac{(\beta - \alpha)(\delta - \gamma)}{(\gamma - \alpha)(\delta - \beta)}}.$$