Biharmonic Curves, Surfaces and Hypersurfaces in Sasakian Space Forms

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Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres


Explicit formulas for biharmonic submanifolds in Sasakian space forms

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Biharmonic hypersurfaces in Sasakian space forms

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Harmonic maps $f : (M, g) \rightarrow (N, h)$ are critical points of the energy

$$E(f) = \frac{1}{2} \int_M |\!df|^2 \, v_g$$

and they are solutions of the Euler-Lagrange equation

$$\tau(f) = \text{trace}_g \nabla df = 0.$$

If $f$ is an isometric immersion, with mean curvature vector field $H$, then:

$$\tau(f) = mH.$$
The bienergy functional (proposed by Eells - Sampson in 1964) is

\[ E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 v_g. \]

Critical points of \( E_2 \) are called biharmonic maps and they are solutions of the Euler-Lagrange equation (Jiang - 1986):

\[ \tau_2(f) = -\Delta^f \tau(\varphi) - \text{trace}_g R^N(df, \tau(f)) df = 0, \]

where \( \Delta^f \) is the Laplacian on sections of \( f^{-1}TN \) and \( R^N \) is the curvature operator on \( N \).
If $\varphi : M \to N$ is an isometric immersion then

$$\tau_2(f) = -m\Delta^f H - m \text{trace } R^N(df, H)df$$

thus $f$ is biharmonic iff

$$\Delta^f H = - \text{trace } R^N(df, H)df.$$
If $f : M \to N(c)$ is an isometric immersion then

$$\tau(f) = mH, \quad \tau_2(\varphi) = -m\Delta^f H + cm^2 H$$

thus $\varphi$ is biharmonic iff

$$\Delta^f H = mcH.$$
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Case $c = 0$ - Chen’s definition

Let $f : M \to \mathbb{R}^n$ be an isometric immersion. Set $f = (f_1, \ldots, f_n)$ and $H = (H_1, \ldots, H_n)$. Then $\Delta^f H = (\Delta H_1, \ldots, \Delta H_n)$, where $\Delta$ is the Beltrami-Laplace operator on $M$, and $\varphi$ is biharmonic iff

$$\Delta^f H = \Delta\left(\frac{-\Delta f}{m}\right) = -\frac{1}{m}\Delta^2 f = 0.$$
Theorem (Jiang - 1986)

Let \( f : (M, g) \rightarrow (N, h) \) be a smooth map. If \( M \) is compact, orientable and \( \text{Riem}^N \leq 0 \) then \( f \) is biharmonic if and only if it is minimal.
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Proposition (Chen - Caddeo, Montaldo, Oniciuc)

If \( c \leq 0 \), there exists no proper biharmonic isometric immersion \( f : M \rightarrow N^3(c) \).
Generalized Chen’s Conjecture

Conjecture (Caddeo, Montaldo, Oniciuc - 2001)

Biharmonic submanifolds of $N^n(c)$, $n > 3$, $c \leq 0$, are minimal.
Generalized Chen’s Conjecture

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*Biharmonic submanifolds of $N^n(c)$, $n > 3$, $c \leq 0$, are minimal.*

Conjecture (Balmuş, Montaldo, Oniciuc - 2007)
*The only proper biharmonic hypersurfaces in $S^{m+1}$ are the open parts of hyperspheres $S^m\left(\frac{1}{\sqrt{2}}\right)$ or of generalized Clifford tori $S^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times S^{m_2}\left(\frac{1}{\sqrt{2}}\right)$, $m_1 + m_2 = m$, $m_1 \neq m_2$.***
Proper-biharmonic curves in spheres

Theorem (Caddeo, Montaldo, Piu - 2001)

The proper-biharmonic curves $\gamma$ of $\mathbb{S}^2$ are circles with radius $\frac{1}{\sqrt{2}}$.

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The proper-biharmonic curves $\gamma$ of $\mathbb{S}^3$ are either circles $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ or geodesics of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ with slope different from $\pm 1$.

Theorem (Caddeo, Montaldo, Oniciuc - 2002)

The proper-biharmonic curves $\gamma$ of $\mathbb{S}^n$, $n > 3$, are those of $\mathbb{S}^3$ up to a totally geodesic embedding.
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*The proper-biharmonic curves $\gamma$ of $S^n$, $n > 3$ are those of $S^3$ up to a totally geodesic embedding.\n
Since odd dimensional spheres $S^{2n+1}$ are Sasakian space forms with constant $\varphi$-sectional curvature 1, the next step is to study the biharmonic submanifolds of Sasakian space forms.
A contact metric structure on a manifold $N^{2m+1}$ is given by $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$ on $N$, $\xi$ is a vector field on $N$, $\eta$ is an 1-form on $N$ and $g$ is a Riemannian metric, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = d\eta(X, Y),$$

for any $X, Y \in C(TN)$.

A contact metric structure $(\phi, \xi, \eta, g)$ is Sasakian if it is normal. The contact distribution of a Sasakian manifold $(N, \phi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$, and an integral curve of the contact distribution is called Legendre curve.
Sasakian space forms

Let \((N, \varphi, \xi, \eta, g)\) be a Sasakian manifold. The sectional curvature of a 2-plane generated by \(X\) and \(\varphi X\), where \(X\) is an unit vector orthogonal to \(\xi\), is called \(\varphi\)-sectional curvature determined by \(X\). A Sasakian manifold with constant \(\varphi\)-sectional curvature \(c\) is called a Sasakian space form and it is denoted by \(N(c)\).
The definition of Frenet curves of osculating order $r$:

Let $(N^n, g)$ be a Riemannian manifold and $\gamma: I \rightarrow N$ a curve parametrized by arc length. Then $\gamma$ is called a Frenet curve of osculating order $r$, $1 \leq r \leq n$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along $\gamma$ such that

$$
E_1 = \gamma' = T, \quad \nabla_T E_1 = \kappa_1 E_2,
$$

$$
\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \ldots,
$$

$$
\nabla_T E_r = -\kappa_{r-1} E_{r-1},
$$

where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on $I$.

A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with $\kappa_1$ constant; a helix of order $r$, $r \geq 3$, is a Frenet curve of osculating order $r$ with $\kappa_1, \ldots, \kappa_{r-1}$ constants; a helix of order 3 is called, simply, helix.
The definition of Frenet curves of osculating order $r$

Definition
Let $(N^n, g)$ be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length. Then $\gamma$ is called a Frenet curve of osculating order $r$, $1 \leq r \leq n$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along $\gamma$ such that $E_1 = \gamma' = T$, $\nabla_T E_1 = \kappa_1 E_2$, $\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3$, $\ldots$, $\nabla_T E_r = -\kappa_{r-1} E_{r-1}$, where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on $I$.

A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with $\kappa_1 =$ constant; a helix of order $r$, $r \geq 3$, is a Frenet curve of osculating order $r$ with $\kappa_1, \ldots, \kappa_{r-1}$ constants; a helix of order 3 is called, simply, helix.
Let \((\mathbb{N}^{2n+1}, \varphi, \xi, \eta, g)\) be a Sasakian space form with constant \(\varphi\)-sectional curvature \(c\) and \(\gamma : I \rightarrow \mathbb{N}\) a Legendre Frenet curve of osculating order \(r\). Then \(\gamma\) is biharmonic iff

\[
\tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T) T
\]

\[
= (-3 \kappa_1 \kappa'_1) E_1 + \left( \kappa''_1 - \kappa_1^3 - \kappa_1 \kappa_2^2 + \frac{(c+3)\kappa_1}{4} \right) E_2
\]

\[
+ (2 \kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 + \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T) \varphi T
\]

\[
= 0.
\]
Proper-biharmonic Legendre curves in Sasakian space forms

Case I ($c = 1$)

Theorem (Fetcu and Oniciuc - 2007)

If $c = 1$ and $n \geq 2$ then $\gamma$ is proper-biharmonic if and only if either $\gamma$ is a circle with $\kappa_1 = 1$ or $\gamma$ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$. 
Proper-biharmonic Legendre curves in Sasakian space forms

Case I \((c = 1)\)

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If \(c = 1\) and \(n \geq 2\) then \(\gamma\) is proper-biharmonic if and only if either \(\gamma\) is a circle with \(\kappa_1 = 1\) or \(\gamma\) is a helix with \(\kappa_1^2 + \kappa_2^2 = 1\).

Case II \((c \neq 1\) and \(\nabla_T T \perp \varphi T)\)

Theorem (Fetcu and Oniciuc - 2007)

Assume that \(c \neq 1\) and \(\nabla_T T \perp \varphi T\). We have

1) if \(c \leq -3\) then \(\gamma\) is biharmonic if and only if it is a geodesic;

2) if \(c > -3\) then \(\gamma\) is proper-biharmonic if and only if either

a) \(n \geq 2\) and \(\gamma\) is a circle with \(\kappa_1^2 = \frac{c+3}{4}\), or

b) \(n \geq 3\) and \(\gamma\) is a helix with \(\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}\).
Case III \((c \neq 1 \text{ and } \nabla_T T \parallel \varphi T)\)

Theorem (Inoguchi - 2004 \((n = 1)\); Fetcu and Oniciuc - 2007)

If \(c \neq 1 \text{ and } \nabla_T T \parallel \varphi T\), then \(\{T, \varphi T, \xi\}\) is the Frenet frame field of \(\gamma\) and we have

1) if \(c < 1\) then \(\gamma\) is biharmonic if and only if it is a geodesic;

2) if \(c > 1\) then \(\gamma\) is proper-biharmonic if and only if it is a helix with \(\kappa_1^2 = c - 1\) \((\text{and } \kappa_2 = 1)\).
Case IV ($c \neq 1$, $n \geq 2$ and $g(E_2, \varphi T)$ is not constant 0, 1 or $-1$)

Theorem (Fetcu and Oniciuc - 2007)

Let $c \neq 1$, $n \geq 2$ and $\gamma$ a Legendre Frenet curve of osculating order $r \geq 4$ such that $g(E_2, \varphi T)$ is not constant 0, 1 or $-1$. We have

a) if $c \leq -3$ then $\gamma$ is biharmonic if and only if it is a geodesic;

b) if $c > -3$ then $\gamma$ is proper-biharmonic if and only if

$\varphi T = \cos \alpha_0 E_2 + \sin \alpha_0 E_4$ and

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant},$$

$$\kappa_1^2 + \kappa_2^2 = \frac{c + 3}{4} + \frac{3(c - 1)}{4} \cos^2 \alpha_0, \quad \kappa_2 \kappa_3 = -\frac{3(c - 1)}{8} \sin 2\alpha_0,$$

where $\alpha_0 \in (0, 2\pi) \setminus \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\}$ is a constant such that

$c + 3 + 3(c - 1) \cos^2 \alpha_0 > 0, \quad 3(c - 1) \sin 2\alpha_0 < 0.$
Theorem (Fetcu and Oniciuc - 2007)

Let \( \gamma : I \to \mathbb{S}^{2n+1}(1) \), \( n \geq 2 \), be a proper-biharmonic Legendre curve parametrized by arc length. Then the equation of \( \gamma \) in the Euclidean space \( \mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle , \rangle) \), is either

\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos \left( \sqrt{2}s \right) e_1 + \frac{1}{\sqrt{2}} \sin \left( \sqrt{2}s \right) e_2 + \frac{1}{\sqrt{2}} e_3
\]

where \( \{e_i, J e_j\} \) are constant unit vectors orthogonal to each other, or

\[
\gamma(s) = \frac{1}{\sqrt{2}} \cos(As)e_1 + \frac{1}{\sqrt{2}} \sin(As)e_2 + \\
\frac{1}{\sqrt{2}} \cos(Bs)e_3 + \frac{1}{\sqrt{2}} \sin(Bs)e_4,
\]
where
\[ A = \sqrt{1 + \kappa_1}, \quad B = \sqrt{1 - \kappa_1}, \quad \kappa_1 \in (0, 1), \]
and \( \{e_i\} \) are constant unit vectors orthogonal to each other, with
\[
\langle e_1, \mathcal{I} e_3 \rangle = \langle e_1, \mathcal{I} e_4 \rangle = \langle e_2, \mathcal{I} e_3 \rangle = \langle e_2, \mathcal{I} e_4 \rangle = 0,
\]
\[
A\langle e_1, \mathcal{I} e_2 \rangle + B\langle e_3, \mathcal{I} e_4 \rangle = 0.
\]
where 

\[ A = \sqrt{1 + \kappa_1}, \quad B = \sqrt{1 - \kappa_1}, \quad \kappa_1 \in (0, 1), \]

and \( \{e_i\} \) are constant unit vectors orthogonal to each other, with

\[ \langle e_1, \mathcal{I} e_3 \rangle = \langle e_1, \mathcal{I} e_4 \rangle = \langle e_2, \mathcal{I} e_3 \rangle = \langle e_2, \mathcal{I} e_4 \rangle = 0, \]

\[ A \langle e_1, \mathcal{I} e_2 \rangle + B \langle e_3, \mathcal{I} e_4 \rangle = 0. \]

We also obtained the explicit equations of proper-biharmonic Legendre curves in odd dimensional spheres endowed with a deformed Sasakian structure, given by Cases II and III of the classification.
Theorem (Fetcu and Oniciuc - 2007)

Let $\gamma$ be a proper-biharmonic Legendre curve in $N^{5}(c)$. Then $c > -3$ and $\gamma$ is a helix of order $r$ with $2 \leq r \leq 5$. 
A method to obtain biharmonic submanifolds in a Sasakian space form

Theorem (Fetcu and Oniciuc - 2007)

Let \((N^{2m+1}, \varphi, \xi, \eta, g)\) be a strictly regular Sasakian space form with constant \(\varphi\)-sectional curvature \(c\) and let \(i : M \rightarrow N\) be an \(r\)-dimensional integral submanifold of \(N\). Consider

\[
F : \tilde{M} = I \times M \rightarrow N, \quad F(t, p) = \phi_t(p) = \phi_p(t),
\]

where \(I = S^1\) or \(I = \mathbb{R}\) and \(\{\phi_t\}_{t \in \mathbb{R}}\) is the flow of the vector field \(\xi\). Then \(F : (\tilde{M}, \tilde{g} = dt^2 + i^* g) \rightarrow N\) is a Riemannian immersion and it is proper-biharmonic if and only if \(M\) is a proper-biharmonic submanifold of \(N\).
The previous Theorem provide a classification result for proper-biharmonic surfaces in a Sasakian space form, which are invariant under the action of the flow of $\xi$. 

\[
\text{Theorem (Fetcu and Oniciuc - 2007)}
\]

Let $M^{2}$ be a surface of $N^{2n+1}(c)$ invariant under the flow of the Reeb vector field $\xi$. Then $M$ is proper-biharmonic if and only if, locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where $\gamma$ is a proper-biharmonic Legendre curve.
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Let $M^2$ be a surface of $N^{2n+1}(c)$ invariant under the flow of the Reeb vector field $\xi$. Then $M$ is proper-biharmonic if and only if, locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where $\gamma$ is a proper-biharmonic Legendre curve.
Let \((N^{2n+1}, \varphi, \xi, \eta, g)\) be a strictly regular Sasakian manifold and \(i: \tilde{M} \rightarrow \tilde{N}\) a submanifold of \(\tilde{N}\). Then \(M = \pi^{-1}(\tilde{M})\) is the Hopf cylinder over \(\tilde{M}\), where \(\pi: M \rightarrow \tilde{N} = N/\xi\) is the Boothby-Wang fibration.
Let \((N^{2n+1}, \varphi, \xi, \eta, g)\) be a strictly regular Sasakian manifold and \(i : \bar{M} \to \bar{N}\) a submanifold of \(\bar{N}\). Then \(M = \pi^{-1}(\bar{M})\) is the Hopf cylinder over \(\bar{M}\), where \(\pi : M \to \bar{N} = N/\xi\) is the Boothby-Wang fibration.

Theorem (Inoguchi - 2004)

Let \(S_{\bar{\gamma}}\) be a Hopf cylinder, where \(\bar{\gamma}\) is a curve in the orbit space of \(N^3(c)\), parametrized by arc length. We have

a) if \(c \leq 1\), then \(S_{\bar{\gamma}}\) is biharmonic if and only if it is minimal;

b) if \(c > 1\), then \(S_{\bar{\gamma}}\) is proper-biharmonic if and only if the curvature \(\bar{\kappa}\) of \(\bar{\gamma}\) is constant \(\bar{\kappa}^2 = c - 1\).
We obtained a geometric characterization of biharmonic Hopf cylinders of any dimension in a Sasakian space form. A special case of our result is the case when $\bar{M}$ is a hypersurface.

**Proposition (Fetcu and Oniciuc - 2008)**

*If $\bar{M}$ is a hypersurface of $\bar{N}$, then $M = \pi^{-1}(\bar{M})$ is biharmonic iff*

$$
\begin{align*}
\Delta^{\perp} H &= \left(-|B|^2 + \frac{c(n+1)+3n-1}{2}\right)H \\
2 \text{trace} A_{\nabla^{\perp} H}(\cdot) + n \text{grad}(|H|^2) &= 0.
\end{align*}
$$
Proposition (Fetcu and Oniciuc - 2008)

If $\bar{M}$ is a hypersurface and $|\bar{H}| = \text{constant} \neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if

$$|B|^2 = \frac{c(n + 1) + 3n - 1}{2}.$$

Proposition (Fetcu and Oniciuc - 2008)

If $|\bar{H}| = \text{constant} \neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if

$$|\bar{B}|^2 = \frac{c(n + 1) + 3n - 5}{2}.$$
From the last result we see that there exist no proper-biharmonic hypersurfaces $M = \pi^{-1}(\tilde{M})$ in $N(c)$ if 
$c \leq \frac{5-3n}{n+1}$, which implies that such hypersurfaces do not exist if $c \leq -3$, whatever the dimension of $N$ is.
Takagi's classification of homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n > 1$

Takagi classified all homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n > 1$, and found five types of such hypersurfaces.

We shall consider $u \in (0, \frac{\pi}{2})$ and $r$ a positive constant given by $\frac{1}{r^2} = \frac{c+3}{4}$.

Theorem (Takagi - 1973)

The geodesic spheres (Type A1) in complex projective space $\mathbb{C}P^n(c + 3)$ have two distinct principal curvatures: $\lambda_2 = \frac{1}{r} \cot u$ of multiplicity $2n - 2$ and $a = \frac{2}{r} \cot 2u$ of multiplicity 1.

Theorem (Takagi - 1973)

The hypersurfaces of Type A2 in complex projective space $\mathbb{C}P^n(c + 3)$ have three distinct principal curvatures: $\lambda_1 = -\frac{1}{r} \tan u$ of multiplicity $2p$, $\lambda_2 = \frac{1}{r} \cot u$ of multiplicity $2q$, and $a = \frac{2}{r} \cot 2u$ of multiplicity 1, where $p > 0$, $q > 0$, and $p + q = n - 1$. 
Biharmonic hypersurfaces in Sasakian space forms with $\varphi$-sectional curvature $c > -3$

Theorem (Fetcu and Oniciuc - 2008)

Let $M = \pi^{-1}(\tilde{M})$ be the Hopf cylinder over $\tilde{M}$.

- If $\tilde{M}$ is of Type A1, then $M$ is proper-biharmonic if and only if either
  - $c = 1$ and $(\tan u)^2 = 1$, or
  - $c \in \left[ \frac{-3n^2 + 2n + 1 + 8\sqrt{2n - 1}}{n^2 + 2n + 5}, +\infty \right) \setminus \{1\}$ and
    $$(\tan u)^2 = n + \frac{2c - 2 \pm \sqrt{c^2(n^2 + 2n + 5) + 2c(3n^2 - 2n - 1) + 9n^2 - 30n + 13}}{c + 3}.$$  

- If $\tilde{M}$ is of Type A2, then $M$ is proper-biharmonic if and only if either
  - $c = 1$, $(\tan u)^2 = 1$ and $p \neq q$, or
  - $c \in \left[ \frac{-3(p - q)^2 - 4n + 4 + 8\sqrt{(2p + 1)(2q + 1)}}{(p - q)^2 + 4n + 4}, +\infty \right) \setminus \{1\}$ and
    $$(\tan u)^2 = \frac{n}{2p + 1} + \frac{2c - 2}{(c + 3)(2p + 1)}.$$


As for the other four types of hypersurfaces we have:

Theorem (Fetcu and Oniciuc - 2008)

There are no proper-biharmonic hypersurfaces $M = \pi^{-1}(\bar{M})$, where $\bar{M}$ is a hypersurface of Type B, C, D or E in complex projective space $\mathbb{C}P^n(c + 3)$. 
The bibliography of biharmonic maps
http://beltrami.sc.unica.it/biharmonic/