Biharmonic Curves, Surfaces and Hypersurfaces in Sasakian Space Forms

Dorel Fetcu and Cezar Oniciuc

"Gh. Asachi" Technical University of Iaşi & "Al.I. Cuza" University of Iaşi

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Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres

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Explicit formulas for biharmonic submanifolds in Sasakian space forms

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Biharmonic hypersurfaces in Sasakian space forms

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The energy functional

Harmonic maps $f : (M,g) \to (N,h)$ are critical points of the energy

$$E(f) = \frac{1}{2} \int_M |\, df \,|^2 \, v_g$$

and they are solutions of the Euler-Lagrange equation

$$\boxed{\tau(f) = \mathrm{trace}_g \nabla df = 0.}$$

If f is an isometric immersion, with mean curvature vector field $\mathbf{H}$, then:

$$\tau(f) = m\mathbf{H}.$$

The bienergy functional

The bienergy functional (proposed by Eells - Sampson in 1964) is

$$E_2(f) = \frac{1}{2}\int_M |\,\tau(f)\,|^2\, v_g.$$

Critical points of E_2 are called biharmonic maps and they are solutions of the Euler-Lagrange equation (Jiang - 1986):

$$\tau_2(f) = -\Delta^f \tau(\varphi) - \mathrm{trace}_g R^N(df, \tau(f))df = 0,$$

where Δ^f is the Laplacian on sections of $f^{-1}TN$ and R^N is the curvature operator on N.

Biharmonic submanifolds

If $\varphi : M \to N$ is an isometric immersion then

$$\tau_2(f) = -m\Delta^f \mathbf{H} - m \,\mathrm{trace}\, R^N(df, \mathbf{H})df$$

thus f is biharmonic iff

$$\boxed{\Delta^f \mathbf{H} = -\,\mathrm{trace}\, R^N(df, \mathbf{H})df.}$$

Biharmonic submanifolds of a space form $N(c)$

If $f : M \to N(c)$ is an isometric immersion then

$$\tau(f) = m\mathbf{H}, \quad \tau_2(\varphi) = -m\Delta^f \mathbf{H} + cm^2\mathbf{H}$$

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$$\boxed{\Delta^f \mathbf{H} = mc\mathbf{H}.}$$

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Case $c = 0$ - Chen's definition
Let $f : M \to \mathbb{R}^n$ be an isometric immersion. Set $f = (f_1, \dots, f_n)$ and $\mathbf{H} = (H_1, \dots, H_n)$. Then $\Delta^f \mathbf{H} = (\Delta H_1, \dots, \Delta H_n)$, where Δ is the Beltrami-Laplace operator on M, and φ is biharmonic iff

$$\Delta^f \mathbf{H} = \Delta(\frac{-\Delta f}{m}) = -\frac{1}{m}\Delta^2 f = 0.$$

Non-existence results

Theorem (Jiang - 1986)

Let $f : (M,g) \to (N,h)$ be a smooth map. If M is compact, orientable and $Riem^N \leq 0$ then f is biharmonic if and only if it is minimal.

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Proposition (Chen - Caddeo, Montaldo, Oniciuc)

If $c \leq 0$*, there exists no proper biharmonic isometric immersion* $f : M \to N^3(c)$*.*

Generalized Chen's Conjecture

Conjecture (Caddeo, Montaldo, Oniciuc - 2001)

Biharmonic submanifolds of $N^n(c)$, $n > 3$, $c \leq 0$, *are minimal.*

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Conjecture (Balmuş, Montaldo, Oniciuc - 2007)

The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ *are the open parts of hyperspheres* $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ *or of generalized Clifford tori* $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Proper-biharmonic curves in spheres

Theorem (Caddeo, Montaldo, Piu - 2001)

The proper-biharmonic curves γ of $\mathbb{S}^2$ are circles with radius $\frac{1}{\sqrt{2}}$.

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The proper-biharmonic curves γ of $\mathbb{S}^3$ are either circles $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ or geodesics of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ with slope different from ± 1.

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Theorem (Caddeo, Montaldo, Oniciuc - 2002)

The proper-biharmonic curves γ of $\mathbb{S}^n$, $n > 3$ are those of $\mathbb{S}^3$ up to a totally geodesic embedding.

Since odd dimensional spheres $\mathbb{S}^{2n+1}$ are Sasakian space forms with constant φ-sectional curvature 1, the next step is to study the biharmonic submanifolds of Sasakian space forms.

Sasakian manifolds

A contact metric structure on a manifold N^{2m+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$ on N, ξ is a vector field on N, η is an 1-form on N and g is a Riemannian metric, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X,Y),$$

for any $X, Y \in C(TN)$.

A contact metric structure (φ, ξ, η, g) is Sasakian if it is normal.

The contact distribution of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$, and an integral curve of the contact distribution is called Legendre curve.

Sasakian space forms

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called φ-sectional curvature determined by X. A Sasakian manifold with constant φ-sectional curvature c is called a Sasakian space form and it is denoted by $N(c)$.

Biharmonic equation for Legendre curves in Sasakian space forms

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The definition of Frenet curves of osculating order r

Definition

Let (N^n, g) be a Riemannian manifold and $\gamma : I \to N$ a curve parametrized by arc length. Then γ is called a Frenet curve of osculating order r, $1 \leq r \leq n$, if there exists orthonormal vector fields $E_1, E_2, ..., E_r$ along γ such that $E_1 = \gamma' = T$, $\nabla_T E_1 = \kappa_1 E_2$, $\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3, ..., \nabla_T E_r = -\kappa_{r-1} E_{r-1}$, where $\kappa_1, ..., \kappa_{r-1}$ are positive functions on I.

A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 with $\kappa_1 = \text{constant}$; a helix of order r, $r \geq 3$, is a Frenet curve of osculating order r with $\kappa_1, ..., \kappa_{r-1}$ constants; a helix of order 3 is called, simply, helix.

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Then γ is biharmonic iff

$$\begin{aligned}
\tau_2(\gamma) = {} & \nabla_T^3 T - R(T, \nabla_T T)T \\
= {} & (-3\kappa_1\kappa_1')E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \tfrac{(c+3)\kappa_1}{4}\right)E_2 \\
& +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 + \tfrac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T \\
= {} & 0.
\end{aligned}$$

Proper-biharmonic Legendre curves in Sasakian space forms

Case I ($c = 1$)

Theorem (Fetcu and Oniciuc - 2007)

If $c = 1$ and $n \geq 2$ then γ is proper-biharmonic if and only if either γ is a circle with $\kappa_1 = 1$ or γ is a helix with $\kappa_1^2 + \kappa_2^2 = 1$.

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Case II ($c \neq 1$ and $\nabla_T T \perp \varphi T$)

Theorem (Fetcu and Oniciuc - 2007)

Assume that $c \neq 1$ and $\nabla_T T \perp \varphi T$. We have
1) if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic;
2) if $c > -3$ then γ is proper-biharmonic if and only if either
a) $n \geq 2$ and γ is a circle with $\kappa_1^2 = \frac{c+3}{4}$, or
b) $n \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}$.

Case III ($c \neq 1$ and $\nabla_T T \parallel \varphi T$)

Theorem (Inoguchi - 2004 ($n = 1$); Fetcu and Oniciuc - 2007)

If $c \neq 1$ and $\nabla_T T \parallel \varphi T$, then $\{T, \varphi T, \xi\}$ is the Frenet frame field of γ and we have

1) if $c < 1$ then γ is biharmonic if and only if it is a geodesic;

2) if $c > 1$ then γ is proper-biharmonic if and only if it is a helix with $\kappa_1^2 = c - 1$ (and $\kappa_2 = 1$).

Case IV ($c \neq 1$, $n \geq 2$ and $g(E_2, \varphi T)$ is not constant $0, 1$ or -1)

Theorem (Fetcu and Oniciuc - 2007)

Let $c \neq 1$, $n \geq 2$ and γ a Legendre Frenet curve of osculating order $r \geq 4$ such that $g(E_2, \varphi T)$ is not constant $0, 1$ or -1. We have
a) if $c \leq -3$ then γ is biharmonic if and only if it is a geodesic;
b) if $c > -3$ then γ is proper-biharmonic if and only if
$\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$ *and*

$$\kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant},$$

$$\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2\alpha_0, \quad \kappa_2\kappa_3 = -\frac{3(c-1)}{8}\sin 2\alpha_0,$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ is a constant such that $c + 3 + 3(c-1)\cos^2\alpha_0 > 0$, $3(c-1)\sin 2\alpha_0 < 0$.

Proper-biharmonic Legendre curves in $\mathbb{S}^{2n+1}(1)$

Theorem (Fetcu and Oniciuc - 2007)

Let $\gamma : I \to \mathbb{S}^{2n+1}(1)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc length. Then the equation of γ in the Euclidean space $\mathbb{E}^{2n+2} = (\mathbb{R}^{2n+2}, \langle , \rangle)$, is either

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)e_1 + \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)e_2 + \frac{1}{\sqrt{2}}e_3$$

where $\{e_i, \mathcal{I}e_j\}$ are constant unit vectors orthogonal to each other, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4,$$

where

$$A = \sqrt{1+\kappa_1}, \quad B = \sqrt{1-\kappa_1}, \quad \kappa_1 \in (0,1),$$

and $\{e_i\}$ are constant unit vectors orthogonal to each other, with

$$\langle e_1, \mathscr{I} e_3 \rangle = \langle e_1, \mathscr{I} e_4 \rangle = \langle e_2, \mathscr{I} e_3 \rangle = \langle e_2, \mathscr{I} e_4 \rangle = 0,$$

$$A\langle e_1, \mathscr{I} e_2 \rangle + B\langle e_3, \mathscr{I} e_4 \rangle = 0.$$

where

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and $\{e_i\}$ are constant unit vectors orthogonal to each other, with

$$\langle e_1,\mathscr{I}e_3\rangle=\langle e_1,\mathscr{I}e_4\rangle=\langle e_2,\mathscr{I}e_3\rangle=\langle e_2,\mathscr{I}e_4\rangle=0,$$

$$A\langle e_1,\mathscr{I}e_2\rangle+B\langle e_3,\mathscr{I}e_4\rangle=0.$$

We also obtained the explicit equations of proper-biharmonic Legendre curves in odd dimensional spheres endowed with a deformed Sasakian structure, given by Cases II and III of the classification.

Proper-biharmonic Legendre curves in $N^5(c)$

Theorem (Fetcu and Oniciuc - 2007)

Let γ be a proper-biharmonic Legendre curve in $N^5(c)$. Then $c > -3$ and γ is a helix of order r with $2 \leq r \leq 5$.

A method to obtain biharmonic submanifolds in a Sasakian space form

Theorem (Fetcu and Oniciuc - 2007)

Let $(N^{2m+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and let $\mathbf{i} : M \to N$ be an r-dimensional integral submanifold of N. Consider

$$F : \widetilde{M} = I \times M \to N, \quad F(t, p) = \phi_t(p) = \phi_p(t),$$

where $I = \mathbb{S}^1$ or $I = \mathbb{R}$ and $\{\phi_t\}_{t \in \mathbb{R}}$ is the flow of the vector field ξ. Then $F : (\widetilde{M}, \widetilde{g} = dt^2 + \mathbf{i}^* g) \to N$ is a Riemannian immersion and it is proper-biharmonic if and only if M is a proper-biharmonic submanifold of N.

The previous Theorem provide a classification result for proper-biharmonic surfaces in a Sasakian space form, which are invariant under the action of the flow of ξ.

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Theorem (Fetcu and Oniciuc - 2007)

Let M^2 be a surface of $N^{2n+1}(c)$ invariant under the flow of the Reeb vector field ξ. Then M is proper-biharmonic if and only if, locally, it is given by $x(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve.

Biharmonic Hopf cylinders in a Sasakian space form

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold and $\mathbf{i} : \bar{M} \to \bar{N}$ a submanifold of $\bar{N}$. Then $M = \pi^{-1}(\bar{M})$ is the Hopf cylinder over $\bar{M}$, where $\pi : M \to \bar{N} = N/\xi$ is the Boothby-Wang fibration.

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Theorem (Inoguchi - 2004)

Let $S_{\bar{\gamma}}$ be a Hopf cylinder, where $\bar{\gamma}$ is a curve in the orbit space of $N^3(c)$, parametrized by arc length. We have
a) if $c \leqslant 1$, then $S_{\bar{\gamma}}$ is biharmonic if and only if it is minimal;
b) if $c > 1$, then $S_{\bar{\gamma}}$ is proper-biharmonic if and only if the curvature $\bar{\kappa}$ of $\bar{\gamma}$ is constant $\bar{\kappa}^2 = c - 1$.

Biharmonic hypersurfaces in a Sasakian space form

We obtained a geometric characterization of biharmonic Hopf cylinders of any dimension in a Sasakian space form. A special case of our result is the case when $\bar{M}$ is a hypersurface.

Proposition (Fetcu and Oniciuc - 2008)

If $\bar{M}$ is a hypersurface of $\bar{N}$, then $M = \pi^{-1}(\bar{M})$ is biharmonic iff

$$\begin{cases} \Delta^{\perp}\mathbf{H} = \left(-|B|^2 + \frac{c(n+1)+3n-1}{2}\right)\mathbf{H} \\ 2\,\mathrm{trace}\,A_{\nabla^{\perp}_{\cdot}\mathbf{H}}(\cdot) + n\,\mathrm{grad}(|\mathbf{H}|^2) = 0. \end{cases}$$

Proposition (Fetcu and Oniciuc - 2008)

If $\overline{M}$ *is a hypersurface and* $|\overline{\mathbf{H}}| = \text{constant} \neq 0$*, then* $M = \pi^{-1}(\overline{M})$ *is proper-biharmonic if and only if*

$$|B|^2 = \frac{c(n+1)+3n-1}{2}.$$

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If $|\overline{\mathbf{H}}| = \text{constant} \neq 0$*, then* $M = \pi^{-1}(\overline{M})$ *is proper-biharmonic if and only if*

$$|\overline{B}|^2 = \frac{c(n+1)+3n-5}{2}.$$

From the last result we see that there exist no proper-biharmonic hypersurfaces $M = \pi^{-1}(\bar{M})$ in $N(c)$ if $c \leq \frac{5-3n}{n+1}$, which implies that such hypersurfaces do not exist if $c \leq -3$, whatever the dimension of N is.

Takagi's classification of homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n > 1$

Takagi classified all homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n > 1$, and found five types of such hypersurfaces.
We shall consider $u \in (0, \frac{\pi}{2})$ and r a positive constant given by $\frac{1}{r^2} = \frac{c+3}{4}$.

Theorem (Takagi - 1973)

The geodesic spheres (Type $A1$) in complex projective space $\mathbb{C}P^n(c+3)$ have two distinct principal curvatures: $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2n-2$ and $a = \frac{2}{r}\cot 2u$ of multiplicity 1.

Theorem (Takagi - 1973)

The hypersurfaces of Type $A2$ in complex projective space $\mathbb{C}P^n(c+3)$ have three distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\tan u$ of multiplicity $2p$, $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2q$, and $a = \frac{2}{r}\cot 2u$ of multiplicity 1, where $p > 0$, $q > 0$, and $p + q = n - 1$.

Biharmonic hypersurfaces in Sasakian space forms with φ-sectional curvature $c > -3$

Theorem (Fetcu and Oniciuc - 2008)

Let $M = \pi^{-1}(\bar{M})$ be the Hopf cylinder over $\bar{M}$.

- *If $\bar{M}$ is of Type $A1$, then M is proper-biharmonic if and only if either*
  - $c = 1$ *and* $(\tan u)^2 = 1$, *or*
  - $c \in \left[\frac{-3n^2+2n+1+8\sqrt{2n-1}}{n^2+2n+5}, +\infty\right) \setminus \{1\}$ *and*

$$(\tan u)^2 = n + \frac{2c - 2 \pm \sqrt{c^2(n^2+2n+5) + 2c(3n^2-2n-1) + 9n^2 - 30n + 13}}{c+3}.$$

- *If $\bar{M}$ is of Type $A2$, then M is proper-biharmonic if and only if either*
  - $c = 1$, $(\tan u)^2 = 1$ *and* $p \neq q$, *or*
  - $c \in \left[\frac{-3(p-q)^2-4n+4+8\sqrt{(2p+1)(2q+1)}}{(p-q)^2+4n+4}, +\infty\right) \setminus \{1\}$ *and*

$$(\tan u)^2 = \frac{n}{2p+1} + \frac{2c-2}{(c+3)(2p+1)}$$

As for the other four types of hypersurfaces we have:

Theorem (Fetcu and Oniciuc - 2008)

There are no proper-biharmonic hypersurfaces $M = \pi^{-1}(\overline{M})$, *where* $\overline{M}$ *is a hypersurface of Type* B, C, D *or* E *in complex projective space* $\mathbb{C}P^n(c+3)$.

Bibliography

The bibliography of biharmonic maps

http://beltrami.sc.unica.it/biharmonic/