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Symmetric spaces of BD.I type and Multicomponent Nonlinear Evolution Equations

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Съдържание

Based on:

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- V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov. *Reductions of N-wave interactions related to low–rank simple Lie algebras. I: $\mathbb{Z}_2$-reductions.* J. Phys. A: Math & Gen. **34**, 9425-9461 (2001).
- V. S. Gerdjikov. *Selected Aspects of Soliton Theory. Constant boundary conditions.* In: Prof. G. Manev's Legacy in Contemporary Aspects of Astronomy, Gravitational and Theoretical Physics Eds.: V. Gerdjikov, M. Tsvetkov, Heron Press Ltd, Sofia, 2005. pp. 277-290. **nlin.SI/0604004**
- G. G. Grahovski, V. S. Gerdjikov, N. A. Kostov, V. A. Atanasov. *New integrable multi-component NLS type equations on symmetric spaces: $\mathbb{Z}_4$ and $\mathbb{Z}_6$ reductions*. In the proceedings of Seventh International conference on Geometry, Integrability and Quantization, June 2–10, 2005, Varna, Bulgaria. Eds. Ivailo Mladenov, Manuel de Leon, Softex, Sofia (2006); pp. 154-175. **nlin.SI/0603066**

- Vladimir S. Gerdjikov, David J. Kaup. *How many types of soliton solutions do we know?*
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1 Nonlinear evolution equations and solitons

Lax representation:

$$[L(\lambda), M(\lambda)] = 0$$

Examples and soliton solutions:

KdV eq. $\quad v_t + v_{xxx} + 6vv_x = 0, \qquad v_{1\mathrm{s}} = \dfrac{2k^2}{\cosh^2\left(k(x - 4k^2 t - x_0)\right)},$

NLS eq. $\quad iu_t + \dfrac{1}{2}u_{xx} + 2|u|^2 u = 0, \qquad u_{1\mathrm{s}} = \dfrac{2\nu e^{i\phi(x,t)}}{\cosh\left(2\nu(x - 2\mu t - x_0)\right)},$

where $\phi(x,t) = \mu x + (\nu^2 - \mu^2)t + \phi_0$.

s-G eq. $\quad \phi_{x,t} + \sin\phi(x,t) = 0, \qquad \phi_{1\mathrm{s}} = 4\arctan\left(e^{2\nu(x-x_0)+t/(2\nu)}\right),$

$$\phi_{\mathrm{br}} = 4\arctan\left(\frac{\nu}{2\mu^2}\frac{\sin(2\mu(x - t - x_0))}{\cosh(2\nu(x + t - x_0))}\right).$$

Spectral aspects and discrete spectrum:

KdV $$L_{\mathrm{KdV}}\Psi_{\mathrm{KdV}} = \left(-\frac{d^2}{dx^2} + v(x,t) - \lambda^2\right)\Psi_{\mathrm{KdV}}(x,t,\lambda) = 0,$$

$v_{1\mathrm{s}} \to \lambda^2 = -k^2$

NLS $$L_{\mathrm{NLS}}\Psi_{\mathrm{NLS}} = \left(i\frac{d}{dx} + q_{\mathrm{NLS}}(x,t) - \lambda\sigma_3\right)\Psi_{\mathrm{NLS}}(x,t,\lambda) = 0,$$

$$q_{\mathrm{NLS}}(x,t) = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \qquad u_{1\mathrm{s}} \to \lambda^{\pm} = \mu \pm i\nu,$$

s-G $$L_{\mathrm{sG}}\Psi_{\mathrm{sG}} = \left(i\frac{d}{dx} + q_{\mathrm{sG}}(x,t) - \lambda\sigma_3\right)\Psi_{\mathrm{sG}}(x,t,\lambda) = 0,$$

$$q_{\mathrm{sG}}(x,t) = \begin{pmatrix} 0 & i\phi \\ i\phi & 0 \end{pmatrix}, \qquad \phi_{1\mathrm{s}} \to \lambda^{\pm} = \pm i\nu, \qquad \phi_{\mathrm{br}} \to \pm\lambda^{\pm} = \pm\mu \pm i\nu,$$

2 N-wave systems related to $sl(n)$, $n \geq 3$

The corresponding Lax operator $L(\lambda) \quad (n = 5)$

$$L \equiv i\partial_x + U(x,t,\lambda) \;=\; i\partial_x + [J, Q(x,t)] - \lambda J,$$

$$M \equiv i\partial_x + U(x,t,\lambda) \;=\; i\partial_t + [I, Q(x,t)] - \lambda I,$$

$$\begin{array}{l} J = \mathrm{diag}\,(J_1, J_2, J_3, J_4, J_5), \\ I = \mathrm{diag}\,(I_1, I_2, I_3, I_4, I_5), \end{array} \qquad Q(x,t) = \begin{pmatrix} 0 & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & 0 & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & 0 & Q_{34} & Q_{35} \\ Q_{41} & Q_{42} & Q_{43} & 0 & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & 0 \end{pmatrix},$$

$$\mathrm{tr}\, J = 0, \qquad \mathrm{tr}\, I = 0, \qquad J_1 > J_2 > J_3 > 0, \qquad 0 > J_4 > J_5. \qquad (1)$$

The N-wave equation:

$$i[J, Q_t] - i[I, Q_x] - [[J, Q], [I, Q(x,t)]] = 0.$$

Generic 1-soliton solution is obtained by the dressing method (Zakharov, Manakov, Shabat (1974)) with the dressing factor $u(x,t,\lambda)$:

$$u(x,t,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P(x,t), \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-},$$

$$q(x) \equiv [J,Q] = \lim_{\lambda\to\infty} \lambda \left(J - u(x,\lambda) J u^{-1}(x,\lambda) \right) = -(\lambda^+ - \lambda^-)[J, P(x)]. \tag{2}$$

P – generic projector $P^2(x,t) = P(x,t)$ with rank $s \geq 1$:

$$P(x,t) = \sum_{a,b=1}^{s} |n_a(x,t)\rangle M_{ab}^{-1} \langle n_b^\dagger(x,t)|, \quad M_{ab}(x,t) = \langle n_b^\dagger(x,t)|n_a(x,t)\rangle,$$

$$|n_a(x,t)\rangle = \chi_0^+(x,t,\lambda^+)|n_{0,a}\rangle, \qquad \langle n_{0,a}|S_0|n_{0,b}\rangle = 0.$$

$|n_k\rangle$, or $P(x,t)$ determine the eigensubspases of L with eigenvalues

$\lambda^{\pm} = \mu \pm i\nu$. For $s = 2$ we have $|n_a\rangle$, $a = 1, 2$ and:

$$P(x,t) = \frac{1}{\det M}\Big(|n_1(x,t)\rangle M_{22}\langle n_1^\dagger(x,t)| - |n_2(x,t)\rangle M_{12}\langle n_1^\dagger(x,t)| - |n_1(x,t)\rangle M_{21}\langle n_2^\dagger(x,t)| + |n_2(x,t)\rangle M_{11}\langle n_2^\dagger(x,t)|\Big),$$

$$\det M(x,t) = M_{11}M_{22} - M_{12}M_{21}, \qquad M_{ab}(x,t) = \langle n_a^\dagger(x,t)|n_b(x,t)\rangle, \tag{3}$$

Rank 1 projectors: $P(x,t) = \frac{|n(x,t)\rangle\langle m(x,t)|}{\langle m(x,t)|n(x,t)\rangle}$

$$|n(x)\rangle = \chi_0^+(x,\lambda^+)|n_0\rangle, \qquad \langle m(x)| = \langle m_0|\hat{\chi}_0^-(x,\lambda^-). \tag{4}$$

For $n = 5$ $|n_0\rangle$ and $\langle m_0|$ are constant 5-component vectors. These the 1-soliton solutions are parametrized by:

1. the discrete eigenvalues $\lambda^{\pm} = \mu \pm i\nu$;
   μ is soliton velocity, ν is the amplitude.

2. the vectors $|n_0\rangle, \langle m_0|$ specify the internal degrees of freedom of the soliton. They can be normalized, say to 1. So maximum 4 independent complex parameters are left.

We have several options that will lead to different types of solitons:
– 1) generic case when: all components of $|n_0\rangle$ are non-vanishing;
– 2) special subcases when one (or more) of these components vanish.
Generic one-soliton solution: take $\chi^{\pm}(x,t,\lambda) = e^{-i\lambda(Jx+It)}$:

$$(P(x,t))_{ks} = \frac{1}{k(x,t)} n_{0,k} m_{0,s} e^{-i(\lambda^+ z_k - \lambda^- z_s)}, \qquad (5)$$

$$k(x,t) = \sum_{p=1}^{n} n_{0,p} m_{0,p} e^{-i(\lambda^+ - \lambda^-) z_p(x,t)}, \qquad (6)$$

$$z_p(x,t) = J_p x + I_p t, \qquad Q^{1s}_{ks} = -(\lambda^+ - \lambda^-)(P(x,t))_{ks}, \qquad (7)$$

i.e. all Q_{ij} are non-trivial waves. $k(x,t)$ may vanish for certain values of x, t. Possible singular solutions.

Impose on $U(x,t,\lambda) = q(x,t) - \lambda J$ the involution

$$KU^\dagger(x,t,\lambda^*)K^{-1} = U(x,t,\lambda), \qquad K = \operatorname{diag}(\epsilon_1, \dots, \epsilon_n), \qquad \epsilon_j = \pm 1 \qquad (8)$$

$$Kq^\dagger(x,t)K^{-1} = q(x,t), \qquad Ku^\dagger(x,t,\lambda^*)K^{-1} = u^{-1}(x,t,\lambda),$$

$$\lambda^+ = (\lambda^-)^* = \mu + i\nu, \qquad \langle m_0| = (K|n_0\rangle)^\dagger.$$

Then the one-soliton solution simplifies to

$$q_{ks}^{1s}(x,t) = -\frac{2i\nu(J_k - J_s)}{k_{\rm red}(x,t)} \epsilon_s n_{0,k} n_{0,s}^* e^{\nu(z_k+z_s)} e^{-i\mu(z_k-z_s)}, \qquad (9)$$

$$k_{\rm red}(x,t) = \sum_{p=1}^{n} \epsilon_p |n_{0,p}|^2 e^{2\nu z_p(x,t)}. \qquad (10)$$

Then reduced soliton will have no singularities only if all ϵ_j are equal.

The analysis of solitons obtained with rank 2 projectors is similar, though more complicated. Note that even with the canonical reduction with $K = \mathbb{1}$ one can not guarantee that $\det M > 0$ for all x and t; so one may encounter singular solitons.

3 Effects of reductions on soliton solutions

The reduction group G_R (Mikhailov, 1978) is a finite group which preserves the Lax representation so that the reduction constraints are automatically compatible with the evolution.

G_R must have two realizations:
i) $G_R \subset \mathrm{Aut}\,\mathfrak{g}$ and
ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair:

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J, \qquad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I, \qquad (11)$$

of the Lax representation:

$$\begin{array}{llll}
1) & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & & C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\
2) & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & & C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\
3) & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & & C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\
4) & C_4(U(\kappa_2(\lambda))) = U(\lambda), & & C_4(V(\kappa_2(\lambda))) = V(\lambda),
\end{array}$$

3.1 N-wave system related to $so(5)$

Impose first a reductions of class 4 that does not affect the spectral parameter. Choose $C_2 = S_0$, $\kappa_2(\lambda) = \lambda$, so

$$S_0(U^T(\lambda))S_0^{-1} + U(\lambda) = 0, \qquad S_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Focus our attention on NLEE related to the $so(5)$ algebra. Thus the N-wave system itself consists of 8 equations. A half of them reads

$$\begin{aligned} i(J_1 - J_2)Q_{10,t}(x,t) - i(I_1 - I_2)Q_{10,x}(x,t) + kQ_{11}(x,t)Q_{\overline{01}}(x,t) = 0, \\ iJ_1Q_{11,t}(x,t) - iI_1Q_{11,x}(x,t) - k(Q_{10}Q_{01} + Q_{12}Q_{\overline{01}})(x,t) = 0, \\ i(J_1 + J_2)Q_{12,t}(x,t) - i(I_1 + I_2)Q_{12,x}(x,t) - kQ_{11}(x,t)Q_{01}(x,t) = 0, \\ iJ_2Q_{01,t}(x,t) - iI_2Q_{01,x}(x,t) + k(Q_{\overline{11}}Q_{12} + Q_{\overline{10}}Q_{11})(x,t) = 0. \end{aligned} \qquad (12)$$

where $k := J_1 I_2 - J_2 I_1$ is a constant describing the wave interaction. The other 4 can be obtained by changing $Q_{kn} \leftrightarrow Q_{\overline{kn}}$. Dressing factor:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda) - 1)\, P(x) + \left(\frac{1}{c(\lambda)} - 1 \right) \overline{P}(x) \in SO(5), \qquad (13)$$

$$\overline{P}(x) = S_0 P^T(x) S_0^{-1}.$$

Generic 1-soliton solution reads

$$Q_{10}(x,t) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle} \left(e^{-i(\lambda^+ z_1 - \lambda^- z_2)} n_{0,1} m_{0,2} + e^{i(\lambda^+ z_2 - \lambda^- z_1)} n_{0,4} m_{0,5} \right),$$

$$Q_{11}(x,t) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle} \left(e^{-i\lambda^+ z_1} n_{0,1} m_{0,3} - e^{-i\lambda^- z_1} n_{0,3} m_{0,5} \right),$$

$$Q_{12}(x,t) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle} \left(e^{-i(\lambda^+ z_1 + \lambda^- z_2)} n_{0,1} m_{0,4} + e^{-i(\lambda^- z_1 + \lambda^+ z_2)} n_{0,2} m_{0,5} \right),$$

$$Q_{01}(x,t) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle} \left(e^{-i\lambda^+ z_2} n_{0,2} m_{0,3} + e^{-i\lambda^- z_2} n_{0,3} m_{0,4} \right),$$

$$\langle m|n\rangle = \sum_{k=1}^{5} e^{-i(\lambda^+ - \lambda^-) z_k} n_{0,k} m_{0,k}, \qquad z_k = J_k x + I_k t, \qquad k = 1, 2.$$

The other 4 field can be formally constructed by doing the following transformation

$$Q_{kn} \leftrightarrow Q_{\overline{kn}}, \qquad e^{-i\lambda^+ z_k} \leftrightarrow e^{i\lambda^- z_k}, \qquad n_{0,j} \leftrightarrow m_{0,j}.$$

A typical $\mathbb{Z}_2$ reduction: $KU^\dagger(\lambda^*)K^{-1} = U(\lambda)$ where $K = \operatorname{diag}(\epsilon_1, \epsilon_2, 1, \epsilon_2, \epsilon_1)$ with $\epsilon_k = \pm 1$.

$$J_k = J_k^*, \quad Q_{\overline{10}} = -\epsilon_1\epsilon_2 Q_{10}^*, \quad Q_{\overline{01}} = -\epsilon_2 Q_{01}^*, \quad Q_{\overline{11}} = -\epsilon_1 Q_{11}^*, \quad Q_{\overline{12}} = -\epsilon_1\epsilon_2 Q_{12}^*.$$

Reduced NLEE is given by 4 equation

$$\begin{aligned}
i(J_1 - J_2)Q_{10,t}(x,t) - i(I_1 - I_2)Q_{10,x}(x,t) - k\epsilon_2 Q_{11}(x,t)Q_{01}^*(x,t) = 0,\\
iJ_1 Q_{11,t}(x,t) - iI_1 Q_{11,x}(x,t) - k(Q_{10}Q_{01} + \epsilon_2 Q_{12}Q_{01}^*)(x,t) = 0,\\
i(J_1 + J_2)Q_{12,t}(x,t) - i(I_1 + I_2)Q_{12,x}(x,t) - kQ_{11}(x,t)Q_{01}(x,t) = 0,\\
iJ_2 Q_{01,t}(x,t) - iI_2 Q_{01,x}(x,t) - k\epsilon_1(Q_{11}^* Q_{12} + \epsilon_2 Q_{10}^* Q_{11})(x,t) = 0.
\end{aligned}$$

Then $\lambda^{\pm} = \mu \pm i\nu$, and $|m\rangle = K|n\rangle^*$ and 1-soliton solution becomes

$$Q_{10}(x,t) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(\epsilon_2 e^{-i(\lambda^+ z_1 - (\lambda^+)^* z_2)} n_{0,1} n_{0,2}^* + \epsilon_1 e^{i(\lambda^+ z_2 - (\lambda^+)^* z_1)} n_{0,4} n_{0,5}^* \right),$$

$$Q_{11}(x,t) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(e^{-i\lambda^+ z_1} n_{0,1} n_{0,3}^* - \epsilon_1 e^{-i(\lambda^+)^* z_1} n_{0,3} n_{0,5}^* \right),$$

$$Q_{12}(x,t) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(\epsilon_2 e^{-i(\lambda^+ z_1 + (\lambda^+)^* z_2)} n_{0,1} n_{0,4}^* + \epsilon_1 e^{-i((\lambda^+)^* z_1 + \lambda^+ z_2)} n_{0,2} n_{0,5}^* \right),$$

$$Q_{01}(x,t) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(e^{-i\lambda^+ z_2} n_{0,2} n_{0,3}^* + \epsilon_2 e^{-i(\lambda^+)^* z_2} n_{0,3} n_{0,4}^* \right),$$

$$\langle n^*|K|n\rangle = \epsilon_1 |n_{0,1}|^2 e^{2\nu z_1} + \epsilon_2 |n_{0,2}|^2 e^{2\nu z_2} + |n_{0,3}|^2 + \epsilon_2 |n_{0,4}|^2 e^{-2\nu z_2} + \epsilon_1 |n_{0,5}|^2 e^{-2\nu z_1},$$

Solitons associated with subalgebras of $so(5)$:

1. Suppose $n_{0,1} = n_{0,5} = 0$. The only nonzero waves are $Q_{01}, Q_{\overline{01}}$ related to the simple root α_2 – a $so(3)$ soliton.

2. Another $sl(2)$ soliton is derived when $n_{0,2} = n_{0,4} = 0$. Then $Q_{11}, Q_{\overline{11}}$ are nonvanishing; the so(3) subalgebra is connected with the root $e_1 = \alpha_1 + \alpha_2$.

3. Let $n_{0,3} = 0$. Then $Q_{10}, Q_{\overline{10}}$ and $Q_{12}, Q_{\overline{12}}$ are nonzero waves. The corresponding subalgebra is $so(3) \oplus so(3) \approx so(4)$.

4. If $n_{0,1}^* = n_{0,5}$, $n_{0,2}^* = n_{0,4}$ and $n_{0,3}^* = n_{0,3}$ then

$$
\begin{aligned}
Q_{10}(x,t) &= \frac{-i\nu}{\Delta_1} \sinh 2\theta_0 \cosh \nu(z_1 + z_2) e^{-i\mu(z_1 - z_2 - \delta_1 + \delta_2)}, \\
Q_{11}(x,t) &= -\frac{2\sqrt{2} i\nu}{\Delta_1} \sinh \theta_0 \sinh \nu z_1 e^{-i\mu(z_1 - \delta_1)}, \\
Q_{12}(x,t) &= \frac{-i\nu}{\Delta_1} \sinh 2\theta_0 \cosh \nu(z_1 - z_2) e^{-i\mu(z_1 + z_2 - \delta_1 - \delta_2)}, \\
Q_{01}(x,t) &= \frac{-2\sqrt{2} i\nu}{\Delta_1} \cosh \theta_0 \cosh \nu z_2 e^{-i\mu(z_2 - \delta_2)}, \\
n_{0,1} &= \frac{n_{0,3}}{\sqrt{2}} \sinh \theta_0 e^{i\mu\delta_1}, \qquad n_{0,2} = \frac{n_{0,3}}{\sqrt{2}} \cosh \theta_0 e^{i\mu\delta_2}, \qquad \theta_0 \in \mathbb{R}, \\
\Delta_1(x,t) &= 2\left(\sinh^2 \theta_0 \sinh^2(\nu z_1) + \cosh^2 \theta_0 \cosh^2(\nu z_2)\right).
\end{aligned}
$$

If $\theta_0 = 0$ then a single wave remains nontrivial:

$$Q_{01}(x,t) = \frac{-\sqrt{2}i\nu}{\cosh \nu z_2} e^{-i\mu(z_2-\delta_2)}.$$

3.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions and Doublet Solitons

An additional $\mathbb{Z}_2$ symmetry:

$$\chi^-(x,\lambda) = K_1 \left((\chi^+)^\dagger(x,\lambda^*) \right)^{-1} K_1^{-1}$$

$$\chi^-(x,\lambda) = K_2 \left((\chi^+)^T(x,-\lambda) \right)^{-1} K_2^{-1}$$

where $K_{1,2} \in SO(5)$ and $[K_1, K_2] = 0$. Also $U(x,\lambda)$ satisfies both symmetry conditions. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced 4-wave system reads

$$\begin{aligned}
(J_1 - J_2)\mathbf{q}_{10,t}(x,t) - (I_1 - I_2)\mathbf{q}_{10,x}(x,t) + k\mathbf{q}_{11}(x,t)\mathbf{q}_{01}(x,t) = 0,\\
J_1\mathbf{q}_{11,t}(x,t) - I_1\mathbf{q}_{11,x}(x,t) + k(\mathbf{q}_{12}(x,t) - \mathbf{q}_{10}(x,t))\mathbf{q}_{01}(x,t) = 0,\\
(J_1 + J_2)\mathbf{q}_{12,t}(x,t) - (I_1 + I_2)\mathbf{q}_{12,x}(x,t) - k\mathbf{q}_{11}(x,t)\mathbf{q}_{01}(x,t) = 0,\\
J_2\mathbf{q}_{01,t}(x,t) - I_2\mathbf{q}_{01,x}(x,t) + k(\mathbf{q}_{10}(x,t) + \mathbf{q}_{12}(x,t))q_{11}(x,t) = 0,
\end{aligned}$$

where $\mathbf{q}_{10}(x,t)$, $\mathbf{q}_{11}(x,t)$, $\mathbf{q}_{12}(x,t)$ and $\mathbf{q}_{01}(x,t)$ are real valued fields.

The dressing factor $u(x,\lambda)$ must be invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.

$$K_1 \left(u^\dagger(x,\lambda^*)\right)^{-1} K_1^{-1} = u(x,\lambda), \tag{14}$$

$$K_2 \left(u^T(x,-\lambda)\right)^{-1} K_2^{-1} = u(x,\lambda). \tag{15}$$

If $K_1 = K_2 = \mathbb{1}$ one way to satisfy both conditions is to choose the poles of $u(x,\lambda)$ at $\lambda^\pm = \pm i\nu$ and $|m(x,t)\rangle = |n(x,t)\rangle = e^{\nu(Jx+It)}|n_0\rangle$ real.

The **doublet solution** becomes

$$\mathbf{q}_{10}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 N_2 \cosh \nu[(J_1+J_2)x + (I_1+I_2)t - \xi_1 - \xi_2],$$

$$\mathbf{q}_{11}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 n_{0,3} \sinh \nu(J_1 x + I_1 t - \xi_1),$$

$$\mathbf{q}_{12}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 N_2 \cosh \nu[(J_1-J_2)x + (I_1-I_2)t - \xi_1 + \xi_2],$$

$$\mathbf{q}_{01}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_2 n_{0,3} \cosh \nu(J_2 x + I_2 t - \xi_2),$$

$$\langle n(x,t)|n(x,t)\rangle = 2N_1^2 \cosh 2\nu(J_1 x + I_1 t - \xi_1) + 2N_2^2 \cosh 2\nu(J_2 x + I_2 t - \xi_2) + n_{0,3}^2,$$

where

$$\xi_1 := \frac{1}{2\nu} \ln \frac{n_{0,5}}{n_{0,1}}, \qquad \xi_2 := \frac{1}{2\nu} \ln \frac{n_{0,4}}{n_{0,2}}, \qquad N_1 = \sqrt{n_{0,1} n_{0,5}}, \qquad N_2 = \sqrt{n_{0,2} n_{0,4}}.$$

3.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions and Quadruplet Solitons

Now the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariance of $u(x,t,\lambda)$ is ensured by adding two more terms:

$$u(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda^+} + \frac{K_1 S A^*(x,t)(K_1 S)^{-1}}{\lambda - (\lambda^+)^*} - \frac{K_2 S A(x,t)(K_2 S)^{-1}}{\lambda + \lambda^+} - \frac{K_1 K_2 A^*(x,t)(K_1 K_2)^{-1}}{\lambda + (\lambda^+)^*}.$$

where $A(x,t) = |X(x,t)\rangle\langle F(x,t)|$ and

$$|F(x,t)\rangle = e^{i\lambda^+(Jx+It)}|F_0\rangle.$$

For $|X(x,t)\rangle$ we get a linear system of equations. Skipping the details we obtain the generic quadruplet solution to the 4-wave system associated with the $\mathbf{B}_2$ algebra

$$\mathbf{q}_{10} = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_1 \cosh(\varphi_1 + \varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu \cosh(\varphi_1^* + \varphi_2) - i\nu \cosh(\varphi_1^* - \varphi_2)\right)\right] N_2$$

$$\mathbf{q}_{11} = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_1 \sinh(\varphi_1) - \frac{im\lambda^+}{\mu\nu} N_1^* \sinh(\varphi_1^*)\right] m_0^3$$
$$\mathbf{q}_{12} = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_1 \cosh(\varphi_1 - \varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu\cosh(\varphi_1^* - \varphi_2) - i\nu\cosh(\varphi_1^* + \varphi_2)\right)\right] N_2$$
$$\mathbf{q}_{01} = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_2 \cosh(\varphi_2) - \frac{im\lambda^{+*}}{\mu\nu} N_2^* \cosh(\varphi_2^*)\right] m_0^3.$$

where

$$a(x,t) = \frac{1}{\mu + i\nu}\left[N_1^2 \cosh 2\varphi_1 + N_2^2 \cosh 2\varphi_2 + \frac{F_{0,3}^2}{2}\right], \quad b(x,t) = \frac{m(x,t)}{i\nu},$$

$$c(x,t) = \frac{m(x,t)}{\mu}, \quad m(x,t) = |N_1|^2 \cosh(2\mathrm{Re}\,\varphi_1) + |N_2|^2 \cosh(2\mathrm{Re}\,\varphi_2) + \frac{|m_0^3|^2}{2},$$
$$N_\sigma := \sqrt{m_0^\sigma m_0^{6-\sigma}}, \quad \varphi_\sigma(x,t) := i\lambda^+(J_\sigma x + I_\sigma t) + \frac{1}{2}\log\frac{m_0^\sigma}{m_0^{6-\sigma}}, \quad \sigma = 1, 2.$$

Other **inequivalent** reductions: we can use automorphisms $\tilde{K}_1$ and/or $\tilde{K}_2$ taking values in the Weyl group.

4 MNLS eqs on BD.I-symmetric spaces

These symmetric spaces are $SO(n+2)/SO(n)\times SO(2)$.

$$L\psi = \left(i\frac{d\psi}{dx} + U(x,t,\lambda) \right) \psi(x,t,\lambda) = 0,$$

$$U(x,\lambda) = q(x,t) - \lambda J, \qquad q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Typical reduction for $n=3$ with $K_1 = \mathrm{diag}\,(\epsilon_1, \epsilon_2, 1, \epsilon_2, \epsilon_1)$, $\epsilon_{1,2}^2 = 1$;

$$p_2 = \epsilon_1\epsilon_2 q_2^*, \qquad p_3 = \epsilon_1 q_3^*, \qquad p_4 = \epsilon_1\epsilon_2 q_4^*;$$

gives a 3-component system of NLS equation

$$\begin{aligned} &iq_{2,t} \;+\; q_{2,xx} + 2\epsilon_1(\epsilon_2|q_2|^2 + |q_3|^2)q_2 + \epsilon_1\epsilon_2 q_3^2 q_4^* = 0, \\ &iq_{3,t} \;+\; q_{3,xx} + 2\epsilon_1 q_2 q_4 q_3^* + \epsilon_1(2\epsilon_2|q_2|^2 + 2\epsilon_2|q_4|^2 + |q_3|^2)q_3 = 0, \\ &iq_{4,t} \;+\; q_{4,xx} + 2\epsilon_1(\epsilon_2|q_4|^2 + |q_3|^2)q_4 + \epsilon_1\epsilon_2 q_3^2 q_2^* = 0, \end{aligned}$$

Its soliton solution is given by

$$
\begin{aligned}
q_2 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(\epsilon_2 e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + \epsilon_1 e^{-\nu(x-ut)} n_{0,4} n_{0,5}^* \right), \\
q_3 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* - \epsilon_1 e^{-\nu(x-ut)} n_{0,3} n_{0,5}^* \right), \\
q_4 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(\epsilon_2 e^{\nu(x-ut)} n_{0,1} n_{0,4}^* + \epsilon_1 e^{-\nu(x-ut)} n_{0,2} n_{0,5}^* \right), \\
\Delta &= \epsilon_1 e^{2\nu(x-ut)} |n_{0,1}|^2 + \epsilon_2 \left(|n_{0,2}|^2 + |n_{0,4}|^2 \right) + |n_{0,3}|^2 + \epsilon_1 e^{-2\nu(x-ut)} |n_{0,5}|^2, \\
v &= \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu, .
\end{aligned}
$$

A second reduction via a Weyl reflection S_{e_2}:

$$
K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}
$$

gives

$$p_2 = q_4^*, \qquad p_3 = -q_3^*, \qquad p_4 = q_2^*.$$

and gives rise to another inequivalent system of 3 NLS equations

$$\begin{aligned}
iq_{2,t} \;+\; q_{2,xx} + 2(q_2 q_4^* - |q_3|^2)q_2 + q_3^2 q_2^* &= 0, \\
iq_{3,t} \;+\; q_{3,xx} - 2q_2 q_4 q_3^* + (2q_2 q_4^* + 2q_4 q_2^* - |q_3|^2)q_3 &= 0, \\
iq_{4,t} \;+\; q_{4,xx} + 2(q_4 q_2^* - |q_3|^2)q_4 + q_3^2 q_4^* &= 0.
\end{aligned}$$

Then we have the following one soliton solution

$$\begin{aligned}
q_2 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,4}^* + e^{-\nu(x-ut)} n_{0,4} n_{0,5}^* \right), \\
q_3 &= \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* + e^{-\nu(x-ut)} n_{0,3} n_{0,5}^* \right), \\
q_4 &= \frac{-2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + e^{-\nu(x-ut)} n_{0,2} n_{0,5}^* \right), \\
\Delta &= e^{2\nu(x-ut)} |n_{0,1}|^2 + \left(n_{0,2} n_{0,4}^* + n_{0,2}^* n_{0,4} \right) - |n_{0,3}|^2 + e^{-2\nu(x-ut)} |n_{0,5}|^2.
\end{aligned}$$

Next we consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction, which is a combination of reductions with K_1 and K_2. This is possible only for $\epsilon_1 = -1$. Then

$$p_{2,4} = -\epsilon_2 q_{2,4}^*, \qquad q_2 = -\epsilon_2 q_4, \qquad p_3 = -q_3^*. \tag{17}$$

and we obtain the following system of two equations

$$iq_{2,t} + q_{2,xx} - 2(\epsilon_2 |q_2|^2 + |q_3|^2)q_2 + q_3^2 q_2^* = 0, \tag{18}$$
$$iq_{3,t} + q_{3,xx} - (4\epsilon_2 |q_2|^* + |q_3|^2)q_3 + 2\epsilon_2 (q_2)^2 q_3^* = 0. \tag{19}$$

and its one soliton solution takes the form

$$q_2 = \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \epsilon_2 \left(e^{\nu(x-ut)} n_{0,1} n_{0,2}^* + e^{-\nu(x-ut)} n_{0,2} n_{0,5}^* \right), \tag{20}$$
$$q_3 = \frac{2i\nu}{\Delta} e^{-i\mu(x-vt)} \left(e^{\nu(x-ut)} n_{0,1} n_{0,3}^* + e^{-\nu(x-ut)} n_{0,3} n_{0,5}^* \right), \tag{21}$$
$$\Delta = e^{2\nu(x-ut)} |n_{0,1}|^2 - 2\epsilon_2 |n_{0,2}|^2 - |n_{0,3}|^2 + e^{-2\nu(x-ut)} |n_{0,5}|^2, \tag{22}$$
$$v = \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu. \tag{23}$$

5 MMKdV eqs on BD.I-symmetric spaces

The M-operator for the MMKdV equations takes the form

$$\begin{aligned} M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - \lambda^3 J)\psi(x,t,\lambda), \\ V_2(x,t) &= q(x,t), \qquad V_1(x,t) = i\mathrm{ad}_J^{-1}\partial_x q + \frac{1}{2}\left[\mathrm{ad}_J^{-1} q, q(x,t)\right], \quad (24) \\ V_0(x,t) &= -\partial_{xx}^2 q + \frac{1}{2}\left[\mathrm{ad}_J^{-1} q, \left[\mathrm{ad}_J^{-1} q, q(x,t)\right]\right] + i\left[\partial_x q, q\right], \end{aligned}$$

It will be convenient to introduce the following notations for the n-component vectors

$$\vec{q} = (q_1, \dots, q_n)^T, \qquad \vec{p} = (p_1, \dots, p_n)^T,$$

and also the matrices S_0 and s_0

$$S_0 = \sum_{k=1}^{n+2} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (25)$$

The MMKdV equations can be written down in compact form as

$$\partial_t \vec{q} + \partial^3_{xxx}\vec{q} + 3(\vec{p},\vec{q})\partial_x\vec{q} + 3\left(\partial_x\vec{q},\vec{p}\right)\vec{q} - 3\left(\partial_x\vec{q} s_0\vec{q}\right) s_0\vec{p} = 0, \quad (26)$$
$$\partial_t \vec{p} + \partial^3_{xxx}\vec{p} + 3(\vec{p},\vec{q})\partial_x\vec{p} + 3\left(\partial_x\vec{p},\vec{q}\right)\vec{p} - 3\left(\partial_x\vec{p} s_0\vec{p}\right) s_0\vec{q} = 0, \quad (27)$$

Analogously the MNLS eqs. generalizing the vector NLS are:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0,$$
$$i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} + (\vec{p}, s_0\vec{p})s_0\vec{q} = 0,$$

Consider a $\mathbb{Z}_2$ reduction of the type

$$U^\dagger(\lambda^*) = U(\lambda), \qquad \Rightarrow \qquad \vec{p} = \vec{q}^*. \quad (28)$$

Then we obtain the following reduced systems of MMKdV

$$\partial_t \vec{q} + \partial^3_{xxx}\vec{q} + 3|\vec{q}|^2\partial_x\vec{q} + 3\left(\partial_x\vec{q},\vec{q}^*\right)\vec{q} - 3\left(\partial_x\vec{q} s_0\vec{q}\right) s_0\vec{q}^* = 0.$$

and MNLS

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger,\vec{q})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{q}^* = 0,$$

Applications: for $n = 2$ and $n = 3$ describe $F = 1$ and $F = 2$ BEC (Wadati (2006)).

The 1-soliton solution of the MMKdV reads

$$q_k = \frac{-i\nu e^{-i\mu(x-ut-\delta_0)}}{\cosh(2\nu(x-vt-\xi_0)) + \mathcal{C}} \left(e^{\nu(x-vt-\xi_0)} c_k^* + (-1)^k e^{-\nu(x-vt-\xi_0)} c_{n+3-k} \right),$$

$$c_k = \frac{n_{0,k}}{\sqrt{|n_{0,1}||n_{0,n+2}|}}, \qquad k = 2, \ldots, 2r \qquad \mathcal{C} = \sum_{k=2}^{2r} |n_{0,k}|^2 / 2|n_{0,1}||n_{0,n+2}|,$$

$$v = \nu^2 - 3\mu^2, \qquad u = 3\nu^2 - \mu^2, \qquad \delta_0 = \frac{\arg n_{0,1}}{\mu}, \qquad \xi_0 = \frac{1}{2\nu} \ln \frac{|n_{0,n+2}|}{|n_{0,1}|}.$$

provided we have fixed $\arg n_{0,1} = -\arg n_{0,n+2}$ by using the natural $U(1)$ symmetry of the solution.

Consider $\mathbb{Z}_2$ reduction of MKdV related to $so(5)$ with

$$KU^\dagger(-\lambda^*)K^{-1} = U(\lambda), \qquad \Rightarrow \quad Kq^\dagger K^{-1} = q, \qquad KJK^{-1} = -J$$

$$W_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Rightarrow \qquad K = \begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon_1 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 & 0 \\ -\epsilon_1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

is the Weyl reflection with respect to the hyperplane orthogonal to e_1. The following interrelations hold true

$$q_3 = -\epsilon_1\epsilon_2 q_1^*, \qquad q_2 = -\epsilon_1 q_2^*,$$
$$p_3 = -\epsilon_1\epsilon_2 p_1^*, \qquad p_2 = -\epsilon_1 p_2^*.$$

As a consequence of the reduction we have

$$\lambda^+ = -(\lambda^-)^*, \qquad |m\rangle = K|n\rangle^*, \tag{29}$$

or

$$(\lambda^\pm)^* = -\lambda^\pm, \qquad |n\rangle = SK|n\rangle^*, \qquad \langle m| = \langle m|^*(SK)^{-1}. \tag{30}$$

Applying another $\mathbb{Z}_2$ reduction of the type

$$U^T(-\lambda) = -U(\lambda), \qquad \Rightarrow \qquad q^T = -q, \tag{31}$$

we obtain that

$$\lambda^+ = -\lambda^-, \qquad |m\rangle = |n\rangle. \tag{32}$$

The corresponding system of MKdV is

$$q_{2,t} + q_{2,xxx} - 3(q_2 q_3)_x q_3 + 3\epsilon_1\epsilon_2 q_3 q_2^* q_{3,x} - 6q_2^2 q_{2,x} = 0,$$
$$q_{3,t} + q_{3,xxx} + 3\epsilon_1\epsilon_2 |q_2|_x^2 q_3 - 3(q_2 q_3)_x q_2 - 3(q_2^* q_3)_x q_2^* - 3q_3^2 q_{3,x} = 0.$$

and is new to the best of our knowledge. Again two types of solitons.
The **doublet soliton** $\rightarrow$ $\lambda^\pm = \pm i\nu$ and $|n\rangle = SK|n\rangle^*$ and is given by

$$q_2 = \frac{i\nu e^{i\delta_0}}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}} \left(e^{\nu(x-vt-\xi_0)} c_2 + e^{-\nu(x-vt-\xi_0)} c_4 \right),$$
$$q_3 = \frac{2i\nu c_3 e^{i\delta_0} \sinh \nu(x - vt - \xi_0)}{\epsilon_1 \cosh 2\nu(x - vt - \xi_0) + \mathcal{C}}, \qquad \delta_0 = \arg n_{0,1} = \arg n_{0,5} = \frac{l\pi}{2}, \quad l \in \mathbb{Z}$$
$$\mathcal{C} = (2\epsilon_2 \mathrm{Re}\,(n_{0,2} n_{0,4}) + |n_{0,3}|^2)/2|n_{0,1}||n_{0,5}|,$$
$$c_1^* = -\epsilon_1 c_1, \quad c_2^* = -\epsilon_2 c_4, \quad c_3^* = -c_3, \quad c_5^* = -\epsilon_1 c_5, \quad c_k = \frac{n_{0,k}}{\sqrt{|n_{0,1}||n_{0,n+2}|}},$$

q_3 is either real or purely imaginary valued function.

The quadruplet soliton solution:

$$u(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda_0} - \frac{KSA^*(x,t)SK}{\lambda + \lambda_0^*} - \frac{SA(x,t)S}{\lambda + \lambda_0} + \frac{KA^*(x,t)K}{\lambda - \lambda_0^*}. \tag{33}$$

$$q(x,t) = [J, A - KSA^*SK - SAS + KA^*K](x,t). \tag{34}$$

Find the matrix $A(x,t) = XF^T$ – algebraic set of equations. Here X and F are rectangular matrices of rank $s \leq r$ and $\lambda_0 = \mu + i\nu$. It can be checked that

$$F(x,t) = e^{i(\lambda_0 x + \lambda_0^3 t)J} F_0, \qquad F_0 = \text{const}.$$

In the simplest $s = 1$ case for the factor X one can obtain the following

$$X = \frac{a^* F + bSKF^* - cKF^*}{|a|^2 + b^2 - c^2},$$

where $\phi_{\rm R}^{\pm} = \phi_{\rm R} \pm \frac{1}{2} \ln \frac{|F_{0,2}|}{|F_{0,4}|}$, $\qquad \phi_{\rm I}^{\pm} = \phi_{\rm I} \pm \arg F_{0,4}$ and

$$a(x,t) = \frac{|F_{0,1}F_{0,5}|}{\lambda_0}(\cosh 2(\phi_{\rm R} - i\phi_{\rm I}) + \mathcal{C}_a), \quad \mathcal{C}_a = \frac{F_{0,2}^2 + F_{0,3}^2 + F_{0,4}^2}{2|F_{0,1}F_{0,5}|},$$

$$b(x,t) = \frac{i|F_{0,1}F_{0,5}|}{\nu}(\cosh 2\phi_{\rm R} + \mathcal{C}_b), \quad \mathcal{C}_b = \frac{2{\rm Re}\,(F_{0,2}^*F_{0,4}) + |F_{0,3}|^2}{2|F_{0,1}F_{0,5}|},$$

$$c(x,t) = \frac{|F_{0,1}F_{0,5}|}{\mu}(\cos 2\phi_{\rm I} + \mathcal{C}_c), \quad \mathcal{C}_c = \frac{|F_{0,2}|^2 - |F_{0,3}|^2 + |F_{0,4}|^2}{2|F_{0,1}F_{0,5}|},$$

$$\phi_{\rm R} = \nu\left(x - vt - \frac{1}{2\nu}\ln\frac{|F_{0,1}|}{|F_{0,5}|}\right), \quad \phi_{\rm I} = \mu\left(x - ut - \frac{\arg F_{0,5}}{\mu}\right),$$

and $\arg F_{0,1} = -\arg F_{0,5}$. Thus for $\epsilon_1 = \epsilon_2 = 1$ one derives

$$q_2 = \frac{2\sqrt{|F_{0,1}F_{0,2}F_{0,4}F_{0,5}|}}{|a|^2 + b^2 - c^2}\left\{a^* \cosh(\phi_{\rm R}^- - i\phi_{\rm I}^-) - b(\cosh(\phi_{\rm R}^- + i\phi_{\rm I}^+) + \cosh(\phi_{\rm R}^+ - i\phi_{\rm I}^-)\right.$$

$$\left. - \; a\cosh(\phi_{\rm R}^+ + i\phi_{\rm I}^+) + c(\cosh(\phi_{\rm R}^+ + i\phi_{\rm I}^-) - \cosh(\phi_{\rm R}^- - i\phi_{\rm I}^+))\right\},$$

$$q_3 = \frac{2i\sqrt{|F_{0,1}F_{0,5}|}}{|a|^2 + b^2 - c^2}{\rm Im}\left\{(b+c)\sinh(\phi_{\rm R} + i\phi_{\rm I}) - a^*\sinh(\phi_{\rm R} - i\phi_{\rm I})\right\}F_{0,3},$$