6-11 June, 2008, *Varna*

Lax pair for restricted multiple three wave interaction system, quasiperiodic solutions and bi-hamiltonian structure

N. A. Kostov, A. V. Tsiganov

N.A. Kostov, A. V. Tsiganov , Exactly Solvable and Integrable Systems (nlin.SI) arXiv:0705.4363v2 [nlin.SI] 2008

INRNE, BAS

1 Restricted multiple three wave interaction system

Several studies have appeared recently on coupled quadratic nonlinear oscillators

$$i\frac{db_j}{d\xi} + uc_j - \frac{1}{2}\epsilon_j b_j = 0, \tag{1}$$

$$i\frac{dc_j}{d\xi} + u^* b_j + \frac{1}{2}\epsilon_j c_j = 0, \tag{2}$$

$$i\frac{du}{d\xi} + \sum_{j=1}^{n} b_j c_j^* = 0, \tag{3}$$

where ξ is the evolution coordinate and ϵ_j corresponds to the normalized wave number mismatchs. The system (1-3) is introduced to model the growth of a low frequency internal ocean wave by interaction with higher frequency surface waves and is used as a model of plasma turbulence. This system describe triads of waves (b_j, c_j, u), $j = 1, \ldots n$ evolving in ξ

and interacting with each other trough multiple three wave interaction with possible applications in optics.

2 Lax representation

The equations (1-3) can be written as Lax representation

$$\frac{dL}{d\xi} = [M, L], \tag{4}$$

of the following linear system:

$$\frac{d\psi}{d\xi} = M(\xi,\lambda)\psi(\xi,\lambda) \quad L(\xi,\lambda)\psi(\xi,\lambda) = 0, \tag{5}$$

where L, M are 2×2 matrices and have the form

$$L(\xi,\lambda) = \begin{pmatrix} A(\xi,\lambda) & B(\xi,\lambda) \\ C(\xi,\lambda) & D(\xi,\lambda) \end{pmatrix}, \tag{6}$$

$$M(\xi,\lambda) = \begin{pmatrix} -i\lambda/2 & iu \\ u^* & i\lambda/2 \end{pmatrix}. \tag{7}$$

where

$$A(\xi,\lambda) = a(\lambda)\left(-i\frac{\lambda}{2} + \frac{i}{2}\sum_{j=1}^{n}\frac{(c_j c_j^* - b_j b_j^*)}{\lambda - \epsilon_j}\right), \tag{8}$$

$$B(\xi,\lambda) = a(\lambda)\left(iu - i\sum_{j=1}^{n}\frac{b_j c_j^*}{\lambda - \epsilon_j}\right), \tag{9}$$

$$C(\xi,\lambda) = a(\lambda)\left(iu^* - i\sum_{j=1}^{n}\frac{c_j b_j^*}{\lambda - \epsilon_j}\right), \tag{10}$$

where $D(\xi,\lambda) = -A(\xi,\lambda)$ and $a(\lambda) = \prod_{i=1}^{n}(\lambda - \epsilon_i)$. Different Lax representation ($n+2\times n+2$ matrix representation) is derived in [?]. Our Lax representation is (2×2 matrix representation), which allow exact integration of initial system. The Lax representation yields the hyperelliptic curve $K = (\nu,\lambda)$

$$\det(L(\lambda) - \frac{1}{2}\nu \mathrm{I}) = 0, \tag{11}$$

where I is the 2×2 unit matrix. The moduli of the curve (11) generate the integrals of motion $J_0, J_j, K_j, j = 1, \ldots, n$,

$$\nu^2 = A^2(\xi, \lambda) + B(\xi, \lambda)C(\xi, \lambda). \tag{12}$$

The curve (12) can be written in canonical form as

$$\nu^2 = 4 \prod_{j=1}^{2n+2} (\lambda - \lambda_j) = R(\lambda), \tag{13}$$

where $\lambda_j \neq \lambda_k$ are branching points. From (12) and explicit expressions for $A(\xi, \lambda), B(\xi, \lambda), C(\xi, \lambda)$ we obtain

$$\nu^2 = a(\lambda)^2 \left(\lambda^2 - 4iJ_0 + 4i \sum_{j=1}^{n} \frac{K_j}{\lambda - \epsilon_j} - i \sum_{j=1}^{n} \frac{J_j^2}{(\lambda - \epsilon_j)^2} \right), \tag{14}$$

where

$$K_j = iub_j^* c_j + iu^* b_j c_j^* + i\frac{\epsilon_j}{2}(|c_j|^2 - |b_j|^2)$$

$$-i\frac{1}{2}\sum_{k\neq j}\frac{\left((|b_j|^2-|c_j|^2)(|b_k|^2-|c_k|^2)+2b_j^*c_jb_kc_k^*+2b_jc_j^*b_k^*c_k\right)}{\epsilon_j-\epsilon_k}, \quad (15)$$

$$J_0 = i|u|^2+\frac{1}{2}i\sum_j^n(|b_j|^2-|c_j|^2), \quad J_j = i(|b_j|^2+|c_j|^2).$$

Next we develop a method which allows to construct quasi-periodic and periodic solutions of system (1-3). The method is based on the application of spectral theory for self-adjoint one dimensional Dirac equation with quasi-(periodic) finite gap potential $\mathcal{U} = -u$ cf. Eqs. (1,2)

$$i\frac{d\Psi_{1j}}{d\xi} - \mathcal{U}\Psi_{2j} - i\lambda_j\Psi_{1j} = 0, \quad (16)$$

$$i\frac{d\Psi_{2j}}{d\xi} - \mathcal{U}^*\Psi_{1j} + i\lambda_j\Psi_{1j} = 0, \quad (17)$$

with spectral parameter λ and eigenvalues $\lambda_j = i\epsilon_j/2$. The equation (4) is equivalently written as

$$\frac{dA}{d\xi} = iuC - iu^*B, \quad A(\xi,\lambda) = \sum_{j=0}^{n+1} A_{n+1-j}(\xi)\lambda^j, \tag{18}$$

$$\frac{dB}{d\xi} = -i\lambda B - 2iuA, \quad B(\xi,\lambda) = \sum_{j=0}^{n} B_{n-j}(\xi)\lambda^j, \tag{19}$$

$$\frac{dC}{d\xi} = i\lambda C + 2iu^*A, \quad C(\xi,\lambda) = \sum_{j=0}^{n} C_{n-j}(\xi)\lambda^j, \tag{20}$$

or in different form we have

$$A_{j+1,\xi} = iuC_j - iu^*B_j,\ A_0 = 1, A_1 = c_1, \tag{21}$$

$$iB_{j+1} = -B_{j,\xi} - 2iuA_{j+1}, \quad B_0 = -2u, \tag{22}$$

$$iC_{j+1} = C_{j,\xi} - 2iu^*A_{j+1} \quad C_0 = -2u^*, \tag{23}$$

where c_1 is the constant of integration. Differenciating Eq. (18) and using (12) we can obtain

$$BB_{\xi\xi} - \frac{u_\xi}{u} BB_\xi - \frac{1}{2} B_\xi^2 + \left(\frac{\lambda^2}{2} - i\lambda \frac{u_\xi}{u} + |u|^2 \right) B^2 = 2u^2 \nu. \qquad (24)$$

Using (16) the eigenfunctions Ψ_1, Ψ_2 for finite-gap potential $\mathcal{U}$ (Baker-Akhiezer function) have the form

$$\Psi_1(\xi,\lambda) = \left[\mathcal{U}(\xi) \prod_{j=1}^{n} \frac{\lambda - \mu_j(\xi)}{\lambda - \mu_j(0)} \right]^{1/2} \exp\left\{ -i \int_0^\xi \frac{\sqrt{R(\lambda)}}{\prod_{j=1}^{n} (\lambda - \mu(\xi'))} d\xi' \right\}.$$

$$\Psi_2(\xi,\lambda) = \frac{i}{\mathcal{U}(\xi)} \left(\frac{d\Psi_1(\xi,\lambda)}{d\xi} - \lambda \Psi_1(\xi,\lambda) \right).$$

Finally we obtain the solutions of (1-3) in terms of well known periodic Dirac eigenvalue problem (16,17)

$$b_j = \Psi_1(\xi,\lambda_j), \qquad c_j = \Psi_2(\xi,\lambda_j), \qquad u(\xi) = -\mathcal{U}(\xi), \qquad (25)$$

where $\mu_j(\xi)$ are known functions, auxiliary spectrum of the periodic Dirac eigenvalue problem (16,17).

Let us consider, in particular, a special case of the system (1-3), three wave system, $n = 1$

$$i\frac{dA_1}{d\xi} = \epsilon A_3 A_2^*, \quad i\frac{dA_2}{d\xi} = \epsilon A_1^* A_3, \quad i\frac{dA_3}{d\xi} = \epsilon A_1 A_2. \tag{26}$$

The corresponding elements of Lax matrices are

$$L(\xi,\lambda) = \begin{pmatrix} -i\frac{\lambda}{2} + \frac{i}{2\lambda}\mathcal{A} & -i\epsilon A_1 - i\frac{1}{\lambda}A_3 A_2^* \\ -i\epsilon A_1^* - i\frac{1}{\lambda}A_2 A_3^{*2} & i\frac{\lambda}{2} - \frac{i}{2\lambda}\mathcal{A} \end{pmatrix}, \tag{27}$$

$$M(\xi,\lambda) = \begin{pmatrix} -i\frac{\lambda}{2} & -i\epsilon A_1 \\ -i\epsilon A_1^* & -i\frac{\lambda}{2} \end{pmatrix}, \quad \mathcal{A} = |A_2|^2 - |A_3|^2. \tag{28}$$

To integrate the system (26) we introduce new variable with $\epsilon = 1$

$$\mu = i\frac{1}{A_1}\frac{dA_1}{d\xi} = \frac{A_3 A_2^*}{A_1}, \tag{29}$$

in terms of which our equations can be written as

$$\frac{d\mu}{d\xi} = 2i\sqrt{R(\mu)}, \tag{30}$$

where

$$R(\lambda) = (\frac{1}{4}\lambda^2 - \frac{1}{2}\mathcal{A})^2 - |A_1|^2(\lambda - \mu)(\lambda - \mu^*) = \tag{31}$$

$$\frac{1}{4}\lambda^4 - \alpha_1\lambda^3 + \alpha_2\lambda^2 - \alpha_3\lambda + \alpha_4. \tag{32}$$

The μ variable and $A_j, j = 1, \ldots 3$ obey the equations

$$\alpha_1 = 0, \quad |A_1|^2 - \frac{1}{2}\mathcal{A} = \alpha_2, \tag{33}$$

$$-|A_1|^2(\mu + \mu^*) = \alpha_3, \quad \frac{1}{4}\mathcal{A} - \mu\mu^*|A_1|^2 = \alpha_4. \tag{34}$$

The equation of motion is then

$$\left(\frac{d\mu}{d\xi}\right)^2 = -4\left(\mu^4 + N\mu^2 - H\mu\right) \tag{35}$$

where the system (26) conserves the dimensionless variable N and the Hamiltonian H

$$N = |A_1|^2 - \frac{1}{2}\mathcal{A}, \quad H = A_1 A_2 A_3^* + A_3 A_1^* A_2, \tag{36}$$

Solving Eqs. (34) for μ variable we obtain

$$\mu = \frac{1}{4\nu}\left(H + i\sqrt{P(\nu)}\right), \tag{37}$$

where

$$P(\nu) = 4\nu^3 - 4N\nu^2 + N^2\nu - H^2. \tag{38}$$

We seek the solution A_1 in the following form

$$A_1 = \sqrt{\nu(\xi)}\exp\left(iC\int_0^{\xi}\frac{d\xi'}{\nu(\xi')}\right) = \sqrt{\nu(\xi)}\exp(i\psi(\xi)), \tag{39}$$

where $\nu = |A_1|^2 = \wp(\xi + \omega') + C_1$, $\wp$ is the Weierstrass function, and ω' is half period.Using Eq. (37) and the following equation

$$\frac{d\nu}{d\xi} = -2i\nu(\mu - \mu^*), \tag{40}$$

derived from (29) and three wave equations we obtain

$$\left(\frac{d\nu}{d\xi}\right)^2 = 4\nu^3 - 4N\nu^2 + N^2\nu - H^2, \tag{41}$$

whose solution can be expressed in terms of the Weierstrass elliptic function $\wp$ as

$$\nu = \wp(\xi + \omega') + \frac{N}{3}. \tag{42}$$

Substituting this expression in Eq. (39) we obtain

$$A_1 = \sqrt{\wp(\xi + \omega') + \frac{N}{3}}\, \exp(i\psi(\xi)), \tag{43}$$

where the phase $\psi(\xi)$ is given by

$$\psi(\xi) = \frac{H}{2\wp'(\kappa)} \left(\ln \frac{\sigma(\xi + \omega' - \kappa)}{\sigma(\xi + \omega' + \kappa)} + 2\zeta(\kappa)\xi \right) + \psi_0, \tag{44}$$

where σ, ζ are Weierstrass functions and ψ_0 is initial constant phase.

3 Bi-hamiltonian structure

In this paragraph we will compute r-matrix algebra of restricted multiple three wave interaction system. We note that in Lax representation (82) we remove the function $a(\lambda)$, which is essential for studying Hamiltonian dynamics of restricted multiple three wave interaction system.

Let as consider Lax matrix

$$\mathscr{L}(\lambda) = a^{-1}(\lambda)L(\xi,\lambda) = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & -\mathscr{A} \end{pmatrix} . \tag{45}$$

and introduce standard Poisson bracket, $\{\cdot,\cdot\}_0$

$$\{f,g\}_0 = -i\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial u^*} - \frac{\partial f}{\partial u^*}\frac{\partial g}{\partial u}\right) - i\sum_{j=1}^{n}\left(\frac{\partial f}{\partial b_j}\frac{\partial g}{\partial b_j^*} - \frac{\partial f}{\partial b_j^*}\frac{\partial g}{\partial b_j}\right)$$
$$-i\left(\frac{\partial f}{\partial c_j}\frac{\partial g}{\partial c_j^*} - \frac{\partial f}{\partial c_j^*}\frac{\partial g}{\partial c_j}\right)$$

The entries of $\mathscr{L}$ satisfy to the following well known equations

$$
\begin{aligned}
\{\mathscr{A}(\lambda),\mathscr{A}(\mu)\}_0 &= \{\mathscr{B}(\lambda),\mathscr{B}(\mu)\}_0 = \{\mathscr{C}(\lambda),\mathscr{C}(\mu)\}_0 = 0,\\
\{\mathscr{A}(\lambda),\mathscr{B}(\mu)\}_0 &= \frac{1}{\lambda-\mu}\Big(\mathscr{B}(\mu)-\mathscr{B}(\lambda)\Big),\\
\{\mathscr{A}(\lambda),\mathscr{C}(\mu)\}_0 &= \frac{-1}{\lambda-\mu}\Big(\mathscr{C}(\mu)-\mathscr{C}(\lambda)\Big),\\
\{\mathscr{B}(\lambda),\mathscr{C}(\mu)\}_0 &= \frac{2}{\lambda-\mu}\Big(\mathscr{A}(\mu)-\mathscr{A}(\lambda)\Big),
\end{aligned}
$$

which may be rewritten as linear r-matrix algebra

$$
\{\overset{1}{\mathscr{L}}(\lambda),\overset{2}{\mathscr{L}}(\mu)\}_0 = [r(\lambda-\mu),\overset{1}{\mathscr{L}}(\lambda)+\overset{2}{\mathscr{L}}(\mu)\,]\,, \qquad (46)
$$

here $\overset{1}{\mathscr{L}}(\lambda) = \mathscr{L}(\lambda)\otimes \mathrm{I}$, $\overset{2}{\mathscr{L}}(\mu) = \mathrm{I}\otimes\mathscr{L}(\mu)$ and $r(\lambda-\mu)$ is a classical rational r-matrix:

$$r(\lambda-\mu)=\frac{\Pi}{\lambda-\mu}, \qquad \Pi=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.$$

Remind, that two Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$ are compatible if every linear combination of them is still a Poisson bracket. The corresponding compatible Poisson tensors P_0 and P_1 satisfy to the following equations

$$[\![P_0,P_0]\!]=[\![P_0,P_1]\!]=[\![P_1,P_1]\!]=0, \tag{47}$$

where $[\![.,.]\!]$ is the Schouten bracket. Remind that on a smooth finite-dimensional manifold $\mathscr{M}$ the Schouten bracket of two bivectors X and Y is an antisymmetric contravariant tensor of rank three and its components in local coordinates z_m read

$$[\![X,Y]\!]^{ijk}=-\sum_{m=1}^{dim\,\mathscr{M}}\left(X^{mk}\frac{\partial Y^{ij}}{\partial z_m}+Y^{mk}\frac{\partial X^{ij}}{\partial z_m}+\mathrm{cycle}(i,j,k)\right).$$

The Poisson bracket associated with the Poisson bivector P is equal to

$$\{f(z), g(z)\} = \langle df,\, P\,dg\rangle = \sum_{i,k} P^{ik}(z)\frac{\partial f(z)}{\partial z_i}\frac{\partial g(z)}{\partial z_k}\,. \tag{48}$$

Here df is covector with entries $\partial f/\partial z_i$ and $\langle .,. \rangle$ is a standard vector product.

There are a lot of Poisson brackets $\{.,.\}_1$ compatible with the linear r-matrix bracket (46) similar to the quadratic Sklyanin algebra. Here we consider two examples only.

Proposition 1. *If*

$$\mathscr{A} = \sum_{i=1}^{n} \frac{h_i}{\lambda - \epsilon_i}, \quad \mathscr{B} = 1 + \sum_{i=1}^{n} \frac{e_i}{\lambda - \epsilon_i}, \quad \mathscr{C} = \sum_{i=1}^{n} \frac{f_i}{\lambda - \epsilon_i}, \tag{49}$$

where h_i, e_i, f_i are dynamical variables and ϵ_i are numerical parameters, then the following brackets are compatible with linear r-matrix bracket (46)

$$\{\mathscr{B}(\lambda), \mathscr{B}(\mu)\}_1 = \{\mathscr{A}(\lambda), \mathscr{A}(\mu)\}_1 = 0,$$

$$\begin{aligned}
\{\mathscr{A}(\lambda),\mathscr{B}(\mu)\}_1 &= \frac{1}{\lambda-\mu}\Big(\lambda\mathscr{B}(\mu)-\mu\mathscr{B}(\lambda)\Big)-\mathscr{B}(\lambda)\mathscr{B}(\mu),\\
\{\mathscr{A}(\lambda),\mathscr{C}(\mu)\}_1 &= \frac{-\lambda}{\lambda-\mu}\Big(\mathscr{C}(\mu)-\mathscr{C}(\lambda)\Big)+\mathscr{B}(\lambda)\mathscr{C}(\mu),\\
\{\mathscr{B}(\lambda),\mathscr{C}(\mu)\}_1 &= \frac{2}{\lambda-\mu}\Big(\mu\mathscr{A}(\mu)-\lambda\mathscr{A}(\lambda)\Big)+2\Big(1-\mathscr{B}(\lambda)\Big)\mathscr{A}(\mu),\\
\{\mathscr{C}(\lambda),\mathscr{C}(\mu)\}_1 &= 2\Big(\mathscr{A}(\mu)\mathscr{C}(\lambda)-\mathscr{A}(\lambda)\mathscr{C}(\mu)\Big).
\end{aligned} \tag{50}$$

Proposition 2. *If*

$$\mathscr{A}=h_n\lambda+\sum_{i=1}^{n-1}\frac{h_i}{\lambda-\epsilon_i},\quad \mathscr{B}=e_n+\sum_{i=1}^{n-1}\frac{e_i}{\lambda-\epsilon_i},\quad \mathscr{C}=f_n+\sum_{i=1}^{n-1}\frac{f_i}{\lambda-\epsilon_i} \tag{51}$$

where h_i, e_i, f_i are dynamical variables and ϵ_i are numerical parameters, then the following brackets are compatible with linear r-matrix bracket

(46)

$$
\begin{aligned}
\{\mathscr{B}(\lambda),\mathscr{B}(\mu)\}_1 &= \{\mathscr{A}(\lambda),\mathscr{A}(\mu)\}_1 = 0,\\
\{\mathscr{A}(\lambda),\mathscr{B}(\mu)\}_1 &= \frac{1}{\lambda-\mu}\Big(\lambda\mathscr{B}(\mu)-\mu\mathscr{B}(\lambda)\Big)-\rho_1\mathscr{B}(\lambda)\mathscr{B}(\mu),\\
\{\mathscr{A}(\lambda),\mathscr{C}(\mu)\}_1 &= \frac{-\lambda}{\lambda-\mu}\Big(\mathscr{C}(\mu)-\mathscr{C}(\lambda)\Big)+\rho_1\mathscr{B}(\lambda)\mathscr{C}(\mu)-\rho_2\mathscr{B}(\lambda), \qquad (52)\\
\{\mathscr{B}(\lambda),\mathscr{C}(\mu)\} &= \frac{2}{\lambda-\mu}\Big(\mu\mathscr{A}(\mu)-\lambda\mathscr{A}(\lambda)\Big)+2\Big(1-\rho_1\mathscr{B}(\lambda)\Big)\mathscr{A}(\mu)\\
&- \rho_3\,\mathscr{B}(\lambda)\\
\{\mathscr{C}(\lambda),\mathscr{C}(\mu)\} &= -2\rho_1\Big(\mathscr{A}(\lambda)C(\mu)-\mathscr{A}(\mu)\mathscr{C}(\lambda)\Big)+2\rho_2\Big(\mathscr{A}(\lambda)-\mathscr{A}(\mu)\Big)\\
&+ \rho_3\,\Big(\mathscr{C}(\lambda)-\mathscr{C}(\mu)\Big)
\end{aligned}
$$

In the both cases we can rewrite second Poisson brackets in the following r-matrix form

$$\{\overset{1}{\mathscr{L}}(\lambda), \overset{2}{\mathscr{L}}(\mu)\}_1 = [r_{12}(\lambda,\mu), \overset{1}{T}(\lambda)] - [r_{21}(\lambda,\mu), \overset{2}{T}(\mu)]\,, \qquad (53)$$

where

$$r_{12}(\lambda,\mu) = \begin{pmatrix} \frac{\mu}{\lambda-\mu} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\lambda-\mu} & 0 \\ 0 & \frac{\lambda}{\lambda-\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mu}{\lambda-\mu} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_1 B(\mu) & 0 \\ -\rho_3 & 0 & 0 & 0 \\ \rho_2 - \rho_1 C(\mu) & 0 & 0 & 0 \end{pmatrix} \qquad (54)$$

and

$$r_{21}(\lambda,\mu) = \Pi r_{12}(\mu,\lambda)\Pi.$$

For the Lax matrix $\mathscr{L}$ with entries (49) we have

$$\rho_1 = 1, \qquad \rho_2 = 0, \qquad \rho_3 = 0.$$

Other brackets are equal to zero, for instance

$$\{b_i, b_j\}_1 = \{b_i, c_j^*\} = \{c_i^*, c_j^*\}_1 = 0.$$

Using r-matrix algebras (46) and (53) it is easy to prove that integrals of motion $I_j \in \{J_0, J_i, \dots, J_n, K_1, \dots, K_n\}$ (15) are in the bi-involution with respect to the brackets $\{.,.\}_0$ and $\{.,.\}_1$:

$$\{I_j, I_k\}_0 = \{I_j, I_k\}_1 = 0\,.$$

The one of the main problems is that the main characteristic of the model, such as equation of motion, the Lax matrices and integrals of motion are invariant with respect to conjugation

$$u \leftrightarrow u^*, \qquad b_j \leftrightarrow b_j^*, \qquad c_j \leftrightarrow c_j^*, \tag{55}$$

but the second brackets $\{.,.\}_1$ do not invariant with respect to this transformation.

The second problem is that the second brackets $\{.,.\}_1$ are rational brackets and we have an obvious problem with quantization of these brackets.

4 Basic equations and solutions

We consider coupled nonlinear Schrödinger equations of the form (Manakov model)

$$\begin{aligned} \imath Q_{1t} + Q_{1xx} + \sigma(|Q_1|^2 + |Q_2|^2)Q_1 = 0, \\ \imath Q_{2t} + Q_{2xx} + \sigma(|Q_1|^2 + |Q_2|^2)Q_2 = 0, \end{aligned} \tag{56}$$

where $\sigma = 1$ for focussing case and $\sigma = -1$ for defocussing case. We seek solution of (56), $\sigma = 1$ in the form

$$Q_j = q_j(z)\, \mathrm{e}^{\imath \Theta_j}, \quad j = 1, 2, \tag{57}$$

where $z = x - ct$, $\Theta_j = \Theta_j(z,t)$, with q_j, Θ_j real, where the functions $\Theta_j, j = 1, 2$ behave as

$$\Theta_j = \frac{1}{2}cx + (a_j - \frac{1}{4}c^2)t - \mathcal{C}_j \int_0^z \frac{\mathrm{d}z'}{q_j^2(z')} + \Theta_{j0}.$$

5 Exact solutions of the Manakov system using Lax pairs method

The system (56) can be solved by the inverse scattering transform method and is written as zero-curvature representation form by

$$U_t - V_x + [U, V] = 0, \tag{58}$$

where U, V are 3×3 matrices defined as

$$U = \begin{pmatrix} -i\lambda & \frac{Q_1}{\sqrt{2}} & \frac{Q_2}{\sqrt{2}} \\ -\frac{\sigma}{\sqrt{2}}Q_1^* & i\lambda & 0 \\ -\frac{\sigma}{\sqrt{2}}Q_2^* & 0 & i\lambda \end{pmatrix}, \qquad V = V_0\lambda^2 + V_1\lambda + V_2 \tag{59}$$

and

$$V_0 = \begin{pmatrix} -2i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & 2i \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 & Q_1\sqrt{2} & Q_2\sqrt{2} \\ -\sigma Q_1^*\sqrt{2} & 0 & 0 \\ -\sigma Q_2^*\sqrt{2} & 0 & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} \frac{i}{2}\sigma(|Q_1|^2+|Q_2|^2) & i\frac{\sqrt{2}}{2}Q_{1x} & i\frac{\sqrt{2}}{2}Q_{2x} \\ i\sigma\frac{\sqrt{2}}{2}Q_{1x}^* & -\frac{1}{2}i\sigma|Q_1|^2 & -\frac{i}{2}\sigma Q_1^*Q_2 \\ i\sigma\frac{\sqrt{2}}{2}Q_{2x}^* & -\frac{i}{2}\sigma Q_2^*Q_1 & -\frac{1}{2}i\sigma|Q_2|^2 \end{pmatrix},$$

where λ is the spectral parameter. The necessary Lax pair is

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi. \tag{60}$$

Next we define 3×3 matrix $M(x,t,\lambda)$, which is solution of $M_x = [U, M]$ and

$$L = \begin{pmatrix} A(x,t,\lambda) & B^1(x,t,\lambda) & B^2(x,t,\lambda) \\ C^1(x,t,\lambda) & D^{22}(x,t,\lambda) & D^{23}(x,t,\lambda) \\ C^2(x,t,\lambda) & D^{32}(x,t,\lambda) & D^{33}(x,t,\lambda) \end{pmatrix},$$

or in explicit form

$$A_x = \frac{1}{\sqrt{2}}\left(Q_1C^1 + Q_2C^2 + \sigma Q_1^*B^1 + \sigma Q_2^*B^2\right), \tag{61}$$

$$B_x^1 = -2i\lambda B^1 + (D^{22} - A)\frac{Q_1}{\sqrt{2}} + D^{32}\frac{Q_2}{\sqrt{2}}, \tag{62}$$

$$B_x^2 = -2i\lambda B^2 + (D^{33} - A)\frac{Q_2}{\sqrt{2}} + D^{23}\frac{Q_1}{\sqrt{2}}, \tag{63}$$

$$C_x^1 = 2i\lambda C^1 + (D^{22} - A)\sigma\frac{Q_1^*}{\sqrt{2}} + D^{23}\sigma\frac{Q_2^*}{\sqrt{2}}, \tag{64}$$

$$C_x^2 = 2i\lambda C^2 + (D^{33} - A)\sigma\frac{Q_2^*}{\sqrt{2}} + D^{32}\sigma\frac{Q_1^*}{\sqrt{2}}, \tag{65}$$

$$D_x^{22} = -\frac{1}{\sqrt{2}}(\sigma Q_1^* B^1 + Q_1 C^1), \quad D_x^{32} = -\frac{1}{\sqrt{2}}(\sigma Q_2^* B^1 + Q_1 C^2), \tag{66}$$

$$D_x^{23} = -\frac{1}{\sqrt{2}}(\sigma Q_1^* B^2 + Q_2 C^1), \quad D_x^{33} = -\frac{1}{\sqrt{2}}(\sigma Q_2^* B^2 + Q_1 C^2), \tag{67}$$

Decompose all matrix elements in powers of λ as follows

$$\begin{aligned}
A(x,t,\lambda) &= A_0(x,t)\lambda^2 + A_1(x,t)\lambda + A_2(x,t) \\
B^k(x,t,\lambda) &= B_0^k(x,t)\lambda^2 + B_1^k(x,t)\lambda + B_2^k(x,t), && (68) \\
C^k(x,t,\lambda) &= C_0^k(x,t)\lambda^2 + C_1^k(x,t)\lambda + C_2^k(x,t), \\
D^{mn}(x,t,\lambda) &= D_0^{mn}(x,t)\lambda^2 + D_1^{mn}(x,t)\lambda + D_2^{mn}(x,t), && (69)
\end{aligned}$$

where $k=1,2$ and $m,n=2,3$. After substituting (68)–(69) into system (61)–(67) we derive the follwoing recurrent relations

$$A_{l,x} = \frac{1}{\sqrt{2}}\left(Q_1 C_l^1 + Q_2 C_l^2 + \sigma Q_1^* B_l^1 + \sigma Q_2^* B_l^2\right), \tag{70}$$

$$B_{l,x}^1 = -2iB_{l+1}^1 + (D_l^{22} - A_l)\frac{Q_1}{\sqrt{2}} + D_l^{32}\frac{Q_2}{\sqrt{2}}, \tag{71}$$

$$B_{l,x}^2 = -2iB_{l+1}^2 + (D_l^{33} - A_l)\frac{Q_2}{\sqrt{2}} + D_l^{23}\frac{Q_1}{\sqrt{2}}, \tag{72}$$

$$C_{l,x}^1 = 2iC_{l+1}^1 + (D_l^{22} - A_l)\sigma\frac{Q_1^*}{\sqrt{2}} + D_l^{23}\sigma\frac{Q_2^*}{\sqrt{2}}, \tag{73}$$

$$C_{l,x}^2 = 2iC_{l+1}^2 + (D_l^{33} - A_l)\sigma\frac{Q_2^*}{\sqrt{2}} + D_l^{32}\sigma\frac{Q_1^*}{\sqrt{2}}, \tag{74}$$

$$D_{l,x}^{22} = -\frac{1}{\sqrt{2}}(\sigma Q_1^* B_l^1 + Q_1 C_l^1), \quad D_{l,x}^{32} = -\frac{1}{\sqrt{2}}(\sigma Q_2^* B_l^1 + Q_1 C_l^2), \tag{75}$$

$$D_{l,x}^{23} = -\frac{1}{\sqrt{2}}(\sigma Q_1^* B_l^2 + Q_2 C_l^1), \quad D_{l,x}^{33} = -\frac{1}{\sqrt{2}}(\sigma Q_2^* B_l^2 + Q_1 C_l^2). \tag{76}$$

Choose the initial conditions

$$A_0 = -2i, \quad D_0^{22} = D_0^{33} = 2i, \quad D_0^{23} = D_0^{32} = 0,$$
$$B_1^k = \sqrt{2}Q_k, \quad A_1 = 0, \quad D_1^{mn} = 0.$$

Explicitly, one computes

$$C_1^1 = -\sqrt{2}\sigma Q_1^*, \quad C_1^2 = -\sqrt{2}\sigma Q_2^*$$
$$A_2 = \frac{i}{2}(|Q_1^*|^2 + |Q_2^*|^2), \quad B_2^1 = \frac{i}{\sqrt{2}}Q_{1,x}, \quad B_0^2 = \frac{i}{\sqrt{2}}Q_{2,x},$$
$$C_2^1 = \frac{i}{\sqrt{2}}\sigma Q_{1,x}^*, \quad C_2^2 = \frac{i}{\sqrt{2}}\sigma Q_{2,x}^*, \quad D_2^{22} = -\frac{i}{2}\sigma|Q_1|^2 + ia_1,$$
$$D_2^{23} = -\frac{i}{2}\sigma Q_1^* Q_2, \quad D_2^{33} = -\frac{i}{2}\sigma|Q_2|^2 + ia_2, \quad D_2^{32} = -\frac{i}{2}\sigma Q_2^* Q_1.$$

5.1 Case $\sigma = 1$

The Lax representation yields a trigonal nonhyperelliptic curve $V = (\nu, \lambda) \det(L(\lambda) - \nu \mathbf{I}_3) = 0$ where $\mathbf{I}_3$ is the 3×3 unit matrix. For the second flow the curve is given explicitly by

$$(\nu+2\imath\lambda^2)(\nu-2\imath\lambda^2)^2+(A_0\lambda+B_0)(\nu-2\imath\lambda^2)-\imath C_0-4D_0\lambda^2-E_0\lambda-F_0\nu^2-\imath H_0\lambda^4=0, \tag{77}$$

where

$$\begin{aligned}
A_0 &= \imath\left(\sum_{k=1}^{2} Q_k^* Q_{k,x} - Q_{k,x}^* Q_k\right) \quad B_0 = \tilde{H} - a_1 a_2 \\
\tilde{H} &= \frac{1}{2}\sum_{k=1}^{2}|Q_{k,x}|^2 + \frac{1}{4}\left(\sum_{k=1}^{2}|Q_k|^2\right)^2 - \frac{1}{2}\sum_{k=1}^{2} a_k|Q_k|^2 \\
C_0 &= \frac{1}{4}Q_{1,x}Q_{2,x}^*Q_1^*Q_2 + \frac{1}{4}Q_{2,x}Q_{1,x}^*Q_2^*Q_1 - \frac{1}{4}|Q_{1,x}^*|^2|Q_2^*|^2 - \frac{1}{4}|Q_{2,x}^*|^2|Q_1^*|^2 \\
&- \frac{1}{2}\left(|Q_1|^2+|Q_2|^2\right)\left(a_1a_2 - \frac{1}{2}a_2|Q_1|^2 - \frac{1}{2}a_1|Q_2|^2\right) \\
&+ \frac{1}{2}a_2|Q_{1,x}|^2 + \frac{1}{2}a_1|Q_{2,x}|^2, \\
E_0 &= (Q_{1,x}^*Q_1 - Q_1^*Q_{1,x})a_2 + (Q_{2,x}^*Q_2 - Q_2^*Q_{2,x})a_1, \\
D_0 &= \imath a_1a_2, \quad F_0 = \imath(a_1+a_2), \quad H_0 = -4\imath F_0.
\end{aligned} \tag{78}$$

The series of birational transformations

$$\nu = \nu_0 + 2i\lambda^2,\ \nu_0 = \frac{\frac{1}{2}E_0 - \frac{1}{2}\lambda_1 + \frac{1}{4}\nu_1}{4i\lambda_1^2 - 4iF_0 + D_0},\ \lambda = \nu_1,\ \nu_1 = 4w,\ z = i\lambda_1 \tag{79}$$

yields a hyperelliptic curve of genus two defined by canonical form

$$w^2 = 4z^5 + \sigma_4 z^4 + \sigma_3 z^3 + \sigma_2 z^2 + \sigma_1 z + \sigma_0,$$

where the *moduli* of the curve σ_i are expressible in terms of physical parameters - level of energy $\tilde{H}$ and constants $a_1, a_2, \tilde{C}_1, \tilde{C}_2$ as follows

$$\begin{aligned}
\sigma_4 &= -8(a_1 + a_2),\\
\sigma_3 &= -4\tilde{H} + 4(a_1 + a_2)^2 + 8a_1 a_2,\\
\sigma_2 &= 4\tilde{H}(a_1 + a_2) - 4\tilde{G} - C_1^2 - C_2^2 - 8a_1 a_2(a_1 + a_2),\\
\sigma_1 &= 4\tilde{G}(a_1 + a_2) - 4a_1 a_2 \tilde{H} + 2\tilde{C}_1^2 a_2 + 2\tilde{C}_2^2 a_1 + 4a_1^2 a_2^2,\\
\sigma_0 &= -4a_1 a_2 \tilde{G} - \tilde{C}_1^2 a_2^2 - \tilde{C}_2^2 a_1^2.
\end{aligned}$$

where

$$\tilde{G} = \frac{1}{8}\left(Q_1 Q_{2,x}^* - Q_2^* Q_{1,x}\right)\left(Q_1^* Q_{2,x} - Q_2 Q_{1,x}^*\right)$$

$$
\begin{aligned}
&+ \frac{1}{8}\left(Q_2 Q_{1,x} - Q_1 Q_{2,x}\right)\left(Q_2^* Q_{1,x}^* - Q_1^* Q_{2,x}^*\right) \\
&+ \frac{1}{2}\left(|Q_1|^2 + |Q_2|^2\right)\left(a_1 a_2 - \frac{1}{2} a_2 |Q_1|^2 - \frac{1}{2} a_1 |Q_2|^2\right) \\
&- \frac{1}{2} a_2 |Q_{1,x}|^2 - \frac{1}{2} a_1 |Q_{2,x}|^2, \; \tilde{C}_k^2 = \frac{1}{4}\left|\left(Q_k Q_{k,x}^* - Q_k^* Q_{k,x}\right)\right|^2, \; k = 1, 2.
\end{aligned}
$$

The first transformation in (79) transforms initial *sigular* spectral curve to nonsingular trigonal curve of genus two. The second two transformations in (79) transfom nonsingular curve to canonical Weierstrass form of hyperelliptic curve of genus two.

5.2 Case $\sigma = -1$

The Lax representation yields a trigonal nonhyperelliptic curve $V = (\nu, \lambda)\ \det(L(\lambda) - \nu \mathbf{I}_3) = 0$ where $\mathbf{I}_3$ is the 3×3 unit matrix. For the second flow the curve is given explicitly by

$$
(\nu + 2\imath\lambda^2)(\nu - 2\imath\lambda^2)^2 + (A_0\lambda + B_0)(\nu - 2\imath\lambda^2)
$$

$$-\imath C_0 + 4D_0\lambda^2 - E_0\lambda - F_0\nu^2 - \imath H_0\lambda^4 = 0,$$

where

$$\begin{aligned}
A_0 &= -\imath\left(\sum_{k=1}^{2} Q_k^* Q_{k,x} - Q_{k,x}^* Q_k\right) \quad B_0 = \tilde{H} - a_1 a_2 \\
\tilde{H} &= -\frac{1}{2}\sum_{k=1}^{2}|Q_{k,x}|^2 + \frac{1}{4}\left(\sum_{k=1}^{2}|Q_k|^2\right)^2 + \frac{1}{2}\sum_{k=1}^{2} a_k|Q_k|^2 \\
C_0 &= \frac{1}{4}Q_{1,x}Q_{2,x}^*Q_1^*Q_2 + \frac{1}{4}Q_{2,x}Q_{1,x}^*Q_2^*Q_1 - \frac{1}{4}|Q_{1,x}^*|^2|Q_2^*|^2 - \frac{1}{4}|Q_{2,x}^*|^2|Q_1^*|^2 \\
&+ \frac{1}{2}\left(|Q_1|^2 + |Q_2|^2\right)\left(a_1a_2 + \frac{1}{2}a_2|Q_1|^2 + \frac{1}{2}a_1|Q_2|^2\right) \\
&- \frac{1}{2}a_2|Q_{1,x}|^2 - \frac{1}{2}a_1|Q_{2,x}|^2, \\
E_0 &= -(Q_{1,x}^*Q_1 - Q_1^*Q_{1,x})a_2 - (Q_{2,x}^*Q_2 - Q_2^*Q_{2,x})a_1, \\
D_0 &= \imath a_1 a_2, \quad F_0 = \imath(a_1 + a_2), \quad H_0 = -4\imath F_0.
\end{aligned} \tag{80}$$

The series of birational transformations

$$\nu = \nu_0 + 2i\lambda^2,\ \nu_0 = \frac{\frac{1}{2}E_0 - \frac{1}{2}\lambda_1 + \frac{1}{4}\nu_1}{4i\lambda_1^2 - 4iF_0 + D_0},\ \lambda = \nu_1,\ \nu_1 = 4w,\ z = i\lambda_1,$$

yields a hyperelliptic curve of genus two defined by canonical form

$$w^2 = 4z^5 + \sigma_4 z^4 + \sigma_3 z^3 + \sigma_2 z^2 + \sigma_1 z + \sigma_0, \tag{81}$$

where the *moduli* of the curve σ_i are expressible in terms of physical parameters - level of energy $\tilde{H}$ and constants $a_1, a_2, \tilde{C}_1, \tilde{C}_2$ as follows

$$\begin{aligned}
\sigma_4 &= -8(a_1 + a_2),\\
\sigma_3 &= -4\tilde{H} + 4(a_1 + a_2)^2 + 8a_1 a_2,\\
\sigma_2 &= 4\tilde{H}(a_1 + a_2) - 4\tilde{G} - C_1^2 - C_2^2 - 8a_1 a_2(a_1 + a_2),\\
\sigma_1 &= 4\tilde{G}(a_1 + a_2) - 4a_1 a_2 \tilde{H} + 2\tilde{C}_1^2 a_2 + 2\tilde{C}_2^2 a_1 + 4a_1^2 a_2^2,\\
\sigma_0 &= -4a_1 a_2 \tilde{G} - \tilde{C}_1^2 a_2^2 - \tilde{C}_2^2 a_1^2.
\end{aligned}$$

where

$$\tilde{G} = \frac{1}{8}\left(Q_1 Q_{2,x}^* - Q_2^* Q_{1,x}\right)\left(Q_1^* Q_{2,x} - Q_2 Q_{1,x}^*\right)$$

$$
\begin{aligned}
&+ \frac{1}{8}\left(Q_2 Q_{1,x} - Q_1 Q_{2,x}\right)\left(Q_2^* Q_{1,x}^* - Q_1^* Q_{2,x}^*\right) \\
&- \frac{1}{2}\left(|Q_1|^2 + |Q_2|^2\right)\left(a_1 a_2 + \frac{1}{2}a_2|Q_1|^2 + \frac{1}{2}a_1|Q_2|^2\right) \\
&+ \frac{1}{2}a_2|Q_{1,x}|^2 + \frac{1}{2}a_1|Q_{2,x}|^2,\ \tilde{C}_k^2 = \frac{1}{4}|\left(Q_k Q_{k,x}^* - Q_k^* Q_{k,x}\right)|^2,\ k = 1,2.
\end{aligned}
$$

The first transformation in transforms initial *sigular* spectral curve to nonsingular trigonal curve of genus two. The second two transformations in transfom nonsingular curve to canonical Weierstrass form of hyper-elliptic curve of genus two. The Lax equation for completely integrable systems discussed in the previous section

$$
L_x(\lambda') = [M(\lambda'), L(\lambda')] \ , \tag{82}
$$

with matrices L and M given by

$$
L(\lambda') = \begin{pmatrix} V(x,\lambda') & U(x,\lambda') \\ W(x,\lambda') & -V(x,\lambda') \end{pmatrix} \tag{83}
$$

$$
M(\lambda') = \begin{pmatrix} 0 & 1 \\ Q(x,\lambda') & 0 \end{pmatrix} . \tag{84}
$$

is equivalent to the Garnier system, where $U(x,\lambda'), V(x,\lambda'), W(x,\lambda'), Q(x,\lambda')$ have the form

$$U(x,\lambda') = a(\lambda')\left(1 - \sum_{i=1}^{g} \frac{\xi_i\eta_i}{\lambda' - a_i}\right),$$

$$V(x,\lambda') = -\frac{1}{2}U_x(x,\lambda')$$

$$W(x,\lambda') = a(\lambda')\left(\lambda' + \sum_{i=1}^{n} \xi_i\eta_i + \sum_{i=1}^{n} \frac{\xi_{ix}\eta_{ix}}{\lambda' - a_i}\right),$$

$$Q(x,\lambda') = \lambda' + 2\sum_{i=1}^{n} \xi_i\eta_i.$$

Finally we point out one useful expression, which is easy to derive from Lax representation (82)

$$W(x,\lambda') = U(x,\lambda')Q(x,\lambda') - \frac{1}{2}U(x,\lambda')_{xx}. \tag{85}$$

The Lax representation yields the hyperelliptic curve $K = (\mu', \lambda')$

$$\det(L(\lambda') - \mu' \mathrm{I}) = 0, \tag{86}$$

generating the integrals of motion $H, F^{(i)}, i = 1, \ldots, n$. We have

$$\mu^2 = V^2(x, \lambda') + U(x, \lambda')W(x, \lambda'), \tag{87}$$

From (87) and explicit expressions of $U(x, \lambda'), V(x, \lambda'), W(x, \lambda')$ we obtain

$$\mu^2 = a(\lambda')^2 \left(\lambda' + \sum_{i=1}^{n} \frac{H_i}{\lambda' - a_i} + \frac{1}{4} \sum_{i=1}^{n} \frac{J_i^2}{(\lambda' - a_i)^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{I_i}{\lambda' - a_i} \right), \tag{88}$$

where

$$I_i = \sum_{k \neq i} \frac{(\xi_k \eta_{ix} - \eta_i \xi_{kx})(\eta_k \xi_{ix} - \xi_i \eta_{kx})}{a_k - a_i} + \sum_{k \neq i} \frac{(\xi_i \xi_{kx} - \xi_k \xi_{ix})(\eta_i \eta_{kx} - \eta_k \eta_{ix})}{a_k - a_i},$$

$$H_i = \xi_{ix}\eta_{ix} - a_i\xi_i\eta_i - \xi_i\eta_i\left(\sum_{k=1}^{g}\xi_k\eta_k\right),$$
$$J_i = \xi_{ix}\eta_i - \xi_i\eta_{ix}, \tag{89}$$

and $\sum_{i=1}^{g} H_i$ is the Hamiltonian for Garnier system. Simple reduction $\eta_i = \xi_i^*$ gives us the second flow of stationary vector nonlinear Schrödinger equation, where by $*$ we denote complex conjugation.

$$\mathcal{U}(x,t) = q_1(x)\exp\left\{\imath a_1 t + \imath \mathcal{C}_1 \int\limits_{\cdot}^{x} \frac{\mathrm{d}x}{q_1^2(x)}\right\}, \tag{90}$$
$$\mathcal{V}(x,t) = q_2(x)\exp\left\{\imath a_2 t + \imath \mathcal{C}_2 \int\limits_{\cdot}^{x} \frac{\mathrm{d}x}{q_2^2(x)}\right\},$$

where the functions $q_{1,2}(x)$ are supposed to be real and a_1, a_2, C_1, C_2 are real constants. Substituting (90) we reduce the system to the equations

$$\begin{aligned} \frac{d^2 q_1}{dx^2} + \rho q_1^3 + \chi q_1 q_2^2 - a_1 q_1 - \frac{\mathcal{C}_1^2}{q_1^3} &= 0 \\ \frac{d^2 q_2}{dx^2} + \kappa q_2^3 + \chi q_2 q_1^2 - a_2 q_2 - \frac{\mathcal{C}_2^2}{q_2^3} &= 0. \end{aligned} \tag{91}$$

The system (91) is the natural hamiltonian two-particle system with the hamiltonian of the form

$$\begin{aligned} H &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{4}(\rho q_1^4 + 2\chi q_1^2 q_2^2 + \kappa q_2^4) \\ &- \frac{1}{2}a_1 q_1^2 - \frac{1}{2}a_2 q_2^2 + \frac{1}{2}\frac{\mathcal{C}_1^2}{q_1^2} + \frac{1}{2}\frac{\mathcal{C}_2^2}{q_2^2}, \end{aligned} \tag{92}$$

where $p_1(t) = \mathrm{d}q_i(t)/dt$.

6 Lax representation

The system $1 : 2 : 1$ $(\kappa = \chi = \rho = 1)$ is a completely integrable hamiltonian system

$$\begin{aligned} \frac{d^2q_1}{dx^2} + (q_1^2 + q_2^2)q_1 - a_1q_1 - \frac{\mathcal{C}_1^2}{q_1^3} &= 0, \\ \frac{d^2q_2}{dx^2} + (q_2^2 + q_2^2)q_2 - a_2q_2 - \frac{\mathcal{C}_2^2}{q_2^3} &= 0 \end{aligned} \tag{93}$$

with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + \frac{1}{4}\left(q_1^2 + q_2^2\right)^2 - \frac{1}{2}a_1q_1^2 - \frac{1}{2}a_2q_2^2 + \frac{1}{2}\frac{\mathcal{C}_1^2}{q_1^2} + \frac{1}{2}\frac{\mathcal{C}_2^2}{q_2^2}, \tag{94}$$

where the variables $(q_1, p_1; q_2, p_2)$ are the canonicaly conjugated variables with respect to the standard Poisson bracket, $\{\cdot\, ;\, \cdot\}$.

This system permits the Lax representation :

$$
\begin{aligned}
\frac{dL(\lambda)}{dx} &= [M(\lambda), L(\lambda)], \\
L(\lambda) &= \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q(\lambda) & 0 \end{pmatrix}
\end{aligned} \tag{95}
$$

is equvalent to the (93), where $U(\lambda), W(\lambda), Q(\lambda)$ have the form

$$
\begin{aligned}
U(\lambda) &= -a(\lambda)\left(1 + \frac{1}{2}\frac{q_1^2}{\lambda - a_1} + \frac{1}{2}\frac{q_2^2}{\lambda - a_2}\right), \\
V(\lambda) &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}}U(\lambda), \\
W(\lambda) &= a(\lambda)\left(-\lambda + \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{1}{2}\left(p_1^2 + \frac{\mathcal{C}_1^2}{q_1^2}\right)\frac{1}{\lambda - a_1}\right. \\
&+ \left.\frac{1}{2}\left(p_2^2 + \frac{\mathcal{C}_2^2}{q_2^2}\right)\frac{1}{\lambda - a_2}\right), \\
Q(\lambda) &= \lambda - q_1^2 - q_2^2,
\end{aligned}
$$

where $a(\lambda) = (\lambda - a_1)(\lambda - a_2)$.

The Lax representation yields hyperelliptic curve $V = (\nu, \lambda)$,

$$\det(L(\lambda) - \frac{1}{2}\nu 1_2) = 0,$$

where 1_2 be 2×2 unit matrix and is given explicitly as

$$\begin{aligned} \nu^2 &= 4(\lambda - a_1)(\lambda - a_2)(\lambda^3 - \lambda^2(a_1 + a_2) + \lambda(a_1 a_2 - H) - F) \\ &- C_1^2(\lambda - a_2)^2 - C_2^2(\lambda - a_1)^2, \end{aligned} \tag{96}$$

where H is the hamiltonian (94) and the second independent integral of motion F, $\{F; H\} = 0$ is given as

$$\begin{aligned} F &= \frac{1}{4}(p_1 q_2 - p_2 q_1)^2 + \frac{1}{2}(q_1^2 + q_2^2)(a_1 a_2 - \frac{1}{2}a_2 q_1^2 - \frac{1}{2}a_1 q_2^2) \\ &- \frac{1}{2}p_1^2 a_2 - \frac{1}{2}p_2^2 a_1 - \frac{1}{4}\frac{(2a_2 - q_2^2)\mathfrak{C}_1^2}{q_1^2} - \frac{1}{4}\frac{(2a_1 - q_1^2)\mathfrak{C}_2^2}{q_2^2}. \end{aligned} \tag{97}$$

We remark, that the parameters $\mathfrak{C}_i$ are linked with coordinates of the points $(a_i, \nu(a_i))$ by the formula

$$\mathfrak{C}_i^2 = -\frac{\nu(a_i)^2}{(a_i - a_j)^2}, \quad i, j = 1, 2. \tag{98}$$

Let us write the curve (96) in the form

$$\nu^2 = 4\lambda^5 + \alpha_4\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0, \tag{99}$$

where the *moduli* of the curve α_i are expressible in terms of physical parameters - level of energy H and constants a_1, a_2, $\mathcal{C}_1, \mathcal{C}_2$ as follows

$$\begin{aligned}
\alpha_4 &= -8(a_1 + a_2), \\
\alpha_3 &= -4H + 4(a_1 + a_2)^2 + 8a_1a_2, \\
\alpha_2 &= 4H(a_1 + a_2) - 4F - \mathcal{C}_1^2 - \mathcal{C}_2^2 - 8a_1a_2(a_1 + a_2), \\
\alpha_1 &= 4F(a_1 + a_2) - 4a_1a_2H + 2\mathcal{C}_1^2a_2 + 2\mathcal{C}_2^2a_1 + 4a_1^2a_2^2, \\
\alpha_0 &= -4a_1a_2F - \mathcal{C}_1^2a_2^2 - \mathcal{C}_2^2a_1^2.
\end{aligned}$$

Let us define new coordinates μ_1, μ_2 as zeros of the entry $U(\lambda)$ to the Lax operator. Then

$$q_1^2 = 2\frac{(a_1 - \mu_1)(a_1 - \mu_2)}{a_1 - a_2}, \quad q_2^2 = 2\frac{(a_2 - \mu_1)(a_2 - \mu_2)}{a_2 - a_1}. \tag{100}$$

The definition of μ_1, μ_2 in the combination with the Lax representation comes to the equations

$$\nu_i = V(\mu_i) = -\frac{1}{2}\frac{\partial}{\partial x}U(\mu_i), \quad i = 1, 2, \tag{101}$$

which can be transformed to the equations of the the form

$$u_1 = \int_{a_1}^{\mu_1} \mathrm{d}u_1 + \int_{a_2}^{\mu_2} du_1, \tag{102}$$

$$u_2 = \int_{a_1}^{\mu_1} \mathrm{d}u_2 + \int_{a_2}^{\mu_2} du_2 \tag{103}$$

where $\mathrm{d}u_{1,2}$ denote independent canonical holomorphic differentials

$$\mathrm{d}u_1 = \frac{\mathrm{d}\lambda}{\nu}, \quad \mathrm{d}u_2 = \frac{\lambda \mathrm{d}\lambda}{\nu}. \tag{104}$$

and $u_1 = a, u_2 = 2x + b$ with the constants a, b defining by the initial conditions. The integration of the problem is then reduces to the solving of the *Jacobi inversion problem* associated with the curve, which consist in the expession of the symmetric functions of $(\mu_1, \mu_2, \nu_1, \nu_2)$ as function of two complex variables (u_1, u_2).

7 Exact solutions in terms of Kleinian hyperelliptic functions

In this section we give the trajectories of the system under consideration in terms of Kleinan hyperelliptic functions , being associated with the algebraic curve of genus two (99) which can be also written in the form

$$\nu^2 = 4\prod_{i=0}^{4}(\lambda - \lambda_i), \tag{105}$$

where $\lambda_i \neq \lambda_i$ are branching points. At all real branching points the closed intervals $[\lambda_{2i-1}, \lambda_{2i}], i = 0, \ldots 4$ will be referred further as lacunae
.

Now we are in position to write the solution of the of the system in terms of Kleinian σ-functions and identify the constants in terms of the moduli of the curve. Using (100) the solutions of (93) have the following form in terms of Kleinian functions $\wp_{22}(\boldsymbol{u}), \wp_{12}(\boldsymbol{u})$

$$q_1^2 = 2\frac{a_1^2 - \wp_{22}(\boldsymbol{u})a_1 - \wp_{12}(\boldsymbol{u})}{a_1 - a_2},$$

$$q_2^2 = 2\frac{a_2^2 - \wp_{22}(\boldsymbol{u})a_2 - \wp_{12}(\boldsymbol{u})}{a_2 - a_1}, \tag{106}$$

where the vector $\boldsymbol{u}^T = (a, 2x + b)$.

8 Alternative Theta-functional integration

Let us consider the nonstationary Schrödinger equation with a real smooth finite-gap potential $u(x,t)$

$$(i\partial_t + \partial_x^2 - u(x,t))\psi(x,t,\lambda) = 0, \tag{107}$$

For every finite smooth finite-gap potential the following expansion of $u(x,t)$ in terms of squared eigenfunctions $\psi(x,t,\lambda)$ hold on [?] , $u(x,t) = -\sum_{i=1}^{g} |\alpha_i|^2 |\psi(x,t,p_i)|^2 + C$ where α_i, C are constants parametrized in general by nonhyperelliptic Riemann surface K of genus g and p_i are some fixed points on K. In particular case when K is hyperelliptic Riemann surface we can construct the following special Baker-AKhiezer function $\psi(x,t,\lambda) = \sqrt{\mathfrak{F}(z,\lambda)}\exp(\Theta)$ where

$$\Theta = \frac{1}{2}\imath c x + \imath(\lambda - \frac{1}{4}c^2)t - \frac{1}{2}\left(\int_0^z \frac{\nu(\lambda)}{\mathfrak{F}(z',\lambda)}dz'\right),$$

and $z = x - ct$, $\mathfrak{F}(z,\lambda)$ is the generalised Hermite or the standard Hermite polynomial. A brief computation reveals that the BA function solves Hill's equation

$$\left(\frac{d^2}{dz^2} - u(z)\right)\tilde{\psi} = \lambda\tilde{\psi}, \quad \tilde{\psi}(z,\lambda) = \sqrt{\mathfrak{F}(z,\lambda)}\exp\left(-\frac{1}{2}\int_0^z \frac{\nu(\lambda)}{\mathfrak{F}(z',\lambda)}dz'\right).$$

For the special points $a_i - \Delta$ $i = 1,2$ in closed intervals $[\lambda_{2i-1},\lambda_{2i}]$, $i = 1,2$ the functions $Q_i = \alpha_i\psi(x,t,\lambda = a_i - \Delta)$ are solutions of (56). We undertake the shift of the spectral parameter , $\lambda \longrightarrow \lambda + \Delta$, $\Delta = \frac{2}{5}a_1 + \frac{2}{5}a_2$ to make initial curve compatible with the Lax representation. The final formula for the solutions of the system (56) then reads

$$Q_1(x,t) = \sqrt{2\frac{\mathfrak{F}(x,a_1-\Delta)}{a_1-a_2}}\exp\left\{\imath a_1 t - \frac{1}{2}\nu(a_i-\Delta)\int^x \frac{\mathrm{d}x'}{\mathfrak{F}(x',a_1-\Delta)}\right\},$$

$$Q_2(x,t) = \sqrt{2\frac{\mathcal{F}(x,a_2-\Delta)}{a_2-a_1}}\exp\left\{\imath a_2 t - \frac{1}{2}\nu(a_2-\Delta)\int^x \frac{\mathrm{d}x'}{\mathcal{F}(x',a_2-\Delta)}\right\}. \tag{108}$$

Let

$$q_j(z) = \sqrt{A_j\wp(z+\omega') + B_j}, \tag{109}$$

and using the well known relations

$$\int_o^z \frac{dz'}{\wp(z)-\wp(\tilde{a}_j)} = \frac{1}{\wp'(\tilde{a}_j)}\left(2z\zeta(\tilde{a}_j) + \ln\frac{\sigma(z-\tilde{a}_j)}{\sigma(z+\tilde{a}_j)}\right), \tag{110}$$

and

$$\wp(z+\omega') - \wp(\tilde{a}_j) = -\frac{\sigma(z+\omega'+\tilde{a}_j)\sigma(z+\omega'-\tilde{a}_j)}{\sigma(z+\omega')^2\sigma(\tilde{a}_j)^2}, \tag{111}$$

we obtain

$$Q_1 = \sqrt{-A_1}\frac{\sigma(z+\omega'+\tilde{a}_1)}{\sigma(z+\omega')\sigma(\tilde{a}_1)}\exp\left(\frac{i}{2}cx + i(a_1 - \frac{1}{4}c^2)t - (z+\omega')\zeta(\tilde{a}_1)\right),$$

$$Q_2 = \sqrt{-A_2}\frac{\sigma(z+\omega'+\tilde{a}_2)}{\sigma(z+\omega')\sigma(\tilde{a}_2)} \exp\left(\frac{i}{2}cx + i(a_2 - \frac{1}{4}c^2)t - (z+\omega')\zeta(\tilde{a}_2)\right),$$

where

$$\sum_{j=1}^{2} A_j = -2, \quad a_j = \sum_{k=1}^{2} B_k - \frac{B_j}{A_j},$$

$$\frac{\mathcal{C}_j^2}{A_j} = \frac{i}{2}\sqrt{4\lambda^3 - \lambda g_2 - g_3}_{|\lambda = -\frac{B_j}{A_j}}, \quad \wp(\tilde{a}_j) = -\frac{B_j}{A_j} = \hat{a}_j. \quad j = 1, 2 \tag{112}$$

These solutions have the following interpretation in terms of spectral function

$$\psi(x,t,\lambda) = \sqrt{g(x,t)} \exp\left(\theta(x,t,\lambda) - \frac{i}{2}\sqrt{R(\lambda)}\int_0^z \frac{1}{g(z',\lambda)}dz'\right), \tag{113}$$

where $g(x,t) = \mu(x,t) - \lambda = \wp(z-\omega') - \lambda$, $\theta(x,t) = \frac{1}{2}cx + (\lambda + \sum_{k=1}^{2} B_k - \frac{1}{4}c^2)t$ and λ is the spectral parameter. Our solutions are obtained when λ is given at points $-B_j/A_j$ and $Q_j = \sqrt{A_j}\psi(x,t,-B_j/A_j)$. The dynamics is on the Riemann surface $\nu^2 = R(\lambda) = 4\lambda^3 - \lambda g_2 - g_3$ and the function $\mu(x,t)$ obey $\mu_x = \sqrt{R(\mu)}$.

One special solution of system (56), $\sigma = 1$ is written by

$$q_1 = C_1 \mathrm{cn}(\alpha z, k), \qquad q_2 = C_2 \mathrm{cn}(\alpha z, k), \tag{114}$$

where $\alpha^2 = a_1/(2k^2-1)$, $C_1^2 + C_2^2 = 2\alpha^2 k^2$, $a_1 = a_2 = a$ and in the limit $k \to 1$ we obtain well known *Manakov* soliton solution

$$Q_1 = \frac{\sqrt{2a}\epsilon_1 \exp\left\{i\left(\frac{1}{2}c(x-x_0) + (a - \frac{1}{4}c^2)t\right)\right\}}{\mathrm{ch}(\sqrt{a}(x - x_0 - ct))},$$

$$Q_2 = \frac{\sqrt{2a}\epsilon_2 \exp\left\{i\left(\frac{1}{2}c(x-x_0) + (a - \frac{1}{4}c^2)t\right)\right\}}{\mathrm{ch}(\sqrt{a}(x - x_0 - ct))},$$

where we introduce the following notations

$$|\epsilon_1|^2 + |\epsilon_2|^2 = 1, \quad \zeta_1 = \frac{1}{2}c + i\sqrt{a} = \xi + i\eta, \tag{115}$$

where x_0 is the position of soliton, (ϵ_1, ϵ_2) are the components of polarization vector. One notes that the real part of ζ_1 i.e. $c/2$ gives us the soliton velocity while the imaginary part of ζ_1, i.e. $\sqrt{2a}$ gives the soliton amplitude and width. Next solution of the system (56), $\sigma = -1$ we obtain using the following ansatz

$$q_i(\zeta) = \sqrt{A_i \wp^2(\zeta + \omega') + B_i \wp(\zeta + \omega') + C_i}, \quad i = 1, 2 \tag{116}$$

then we have

$$\sum_{i=1}^{2} A_i = 0, \sum_{i=1}^{2} B_i = -6, \, a_i = \sum_{k=1}^{2} C_k - 3\frac{B_i}{A_i}, \, C_i = \frac{B_i^2}{A_i} - \frac{1}{4} A_i g_2, \tag{117}$$

$$\mathfrak{C}_i^2 = \frac{A_i^2}{2 \cdot 3^3 \cdot 4}(4\lambda^5 + 27\lambda^2 g_3 + 27\lambda g_2^2 - 21\lambda^3 g_2 - 81 g_2 g_3), \tag{118}$$

where $\lambda = -3B_i/A_i$. The paremeters A_i, B_i, C_i, $i = 1, 2$ are expressed in terms of a_i by

$$A_1 = \frac{18}{a_1 - a_2}, \quad A_2 = \frac{18}{a_2 - a_1},$$

$$B_1 = -\frac{6(a_1 - \Delta)}{a_1 - a_2}, \quad B_2 = -\frac{6(a_2 - \Delta)}{a_2 - a_1},$$

$$C_1 = \frac{2(a_1-\Delta)^2}{a_1-a_2} - \frac{9}{2}\frac{g_2}{a_1-a_2}, \quad C_2 = \frac{2(a_2-\Delta)^2}{a_2-a_1} - \frac{9}{2}\frac{g_2}{a_2-a_1},$$

and $\Delta = \frac{2}{5}a_1 + \frac{2}{5}a_2$. One can obtain the special solution

$$q_1 = C\mathrm{sn}(\alpha z, k)\mathrm{dn}(\alpha z, k), \qquad q_2 = C\mathrm{cn}(\alpha z, k)\mathrm{dn}(\alpha z, k), \tag{119}$$

where

$$\alpha^2 = \frac{1}{15}(4a_2-a_1), \quad C^2 = \frac{2}{5}(4a_2-a_1), \quad k^2 = 5\frac{a_2-a_1}{4a_2-a_1}, \tag{120}$$

in the limit $k \to 1$ we have the following soliton solution

$$\begin{aligned} Q_1 &= \sqrt{6a}\epsilon_1 \exp\left\{i\left(\frac{1}{2}c(x-x_0)+(a-\frac{1}{4}c^2)t\right)\right\} \times \\ &\quad \mathrm{th}(\sqrt{a}(x-x_0-ct))\mathrm{sech}(\sqrt{a}(x-x_0-ct)), \\ Q_2 &= \frac{\sqrt{6a}\epsilon_2 \exp\left\{i\left(\frac{1}{2}c(x-x_0)+(4a-\frac{1}{4}c^2)t\right)\right\}}{\mathrm{ch}^2(\sqrt{a}(x-x_0-ct))}, \end{aligned}$$

where we introduce the following notations

$$|\epsilon_1|^2 = |\epsilon_2|^2 = 1, \quad a_1 = a,\ a_2 = 4a_1. \tag{121}$$

Another special solution of (56) , $\sigma = -1$ is written by

$$q_1 = C\left(\frac{1}{3}C_1 - k^2\mathrm{sn}^2(\alpha z, k)\right), \tag{122}$$

$$q_2 = C\mathrm{sn}(\alpha z, k)\mathrm{dn}(\alpha z, k), \tag{123}$$

where

$$\alpha^2 = \frac{1}{15}(a_2^2 - a_1^2), \quad C^2 = -\frac{6}{5}(a_1 + a_2),$$

$$k^2 = \frac{\sqrt{\frac{5}{3}(a_1^2 - a_2^2)} + 2a_1 - 3a_2}{2\sqrt{\frac{5}{3}(a_1^2 - a_2^2)}}, \quad \frac{1}{3}C_1 = \frac{1}{2} - \frac{1}{2}\left(\frac{5}{3}\frac{a_1 - a_2}{a_1 + a_2}\right)^{1/2}.$$

in the limit $k \to 1$ we obtain the following soliton solution

$$Q_1 = \sqrt{\frac{9a}{4}}\epsilon_1 \exp\left\{i\left(\frac{1}{2}c(x - x_0) - (a + \frac{1}{4}c^2)t\right)\right\} \times \left(-\frac{2}{3} + \mathrm{sech}^2(\sqrt{\frac{a}{8}}(x - x_0 - ct)\right),$$

$$Q_2 = \sqrt{\frac{9a}{4}}\epsilon_2 \exp\left\{ i \left(\frac{1}{2}c(x - x_0) - (\frac{7}{8}a + \frac{1}{4}c^2)t \right) \right\}$$

$$\text{th}(\sqrt{\frac{a}{8}}(x - x_0 - ct)\text{sech}(\sqrt{\frac{a}{8}}(x - x_0 - ct)),$$

where $|\epsilon_1|^2 = |\epsilon_2|^2 = 1$, $a_1 = -a$, $a_2 = -\frac{7}{8}a$.