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Algebraic aspects of integrable nonlinear evolution equations with deep reductions

I. Equations on symmetric and homogeneous spaces

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Based on:

- V. S. Gerdjikov. *Algebraic and Analytic Aspects of N-wave Type Equations.* Contemporary Mathematics **301**, 35-68 (2002).

- V. S. Gerdjikov, D. J. Kaup, N. A. Kostov, T. I. Valchev. *On classification of soliton solutions of multicomponent nonlinear evolution equations.*
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- Nikolay Kostov, Vladimir Gerdjikov. *Reductions of multicomponent mKdV equations on symmetric spaces of* **DIII***-type* . SIGMA **4** (2008), paper 029, 30 pages; **ArXiv:0803.1651**.

- V. S. Gerdjikov. *Selected Aspects of Soliton Theory. Constant boundary conditions.* In: Prof. G. Manev's Legacy in Contemporary Aspects of Astronomy, Gravitational and Theoretical Physics Eds.: V. Gerdjikov, M. Tsvetkov, Heron Press Ltd, Sofia, 2005. pp. 277-290. **nlin.SI/0604004**

Съдържание

1 BEC with hyperfine structure

$^{23}\mathrm{Na} \Leftrightarrow F = 1 \qquad ^{87}\mathrm{Rb} \Leftrightarrow F = 2$

see Wadati et al (2004), (2006), (2007); Ohmi & Machida (1998); Kuwamoto et al (2004); Gerdjikov et al (2007), (2008)

The assembly of atoms in the hyperfine state of spin F is described by a normalized spinor wave vector with $2F+1$ components

$$\Phi(x,t) = (\Phi_F(x,t), \Phi_{F-1}(x,t), \dots, \Phi_{-F}(x,t))^T$$

whose components are labeled by the values of $m_F = F, \dots, 1, 0, -1, \dots, -F$.

GPE-equation in the one-dimensional approximation:

$$i\frac{\partial \Phi}{\partial t} = \frac{\delta E_{\mathrm{GP}}[\Phi]}{\delta \Phi^*}. \tag{1}$$

where for $F = 1$ the energy functional is given by:

$$E_{\mathrm{GP}} = \int dx \left\{ \frac{\hbar^2}{2m} |\partial_x \Phi|^2 + \frac{\bar{c}_0 + \bar{c}_2}{2} \left[|\Phi_1|^4 + |\Phi_{-1}|^4 + 2|\Phi_0|^2 (|\Phi_1|^2 + |\Phi_{-1}|^2) \right] \right.$$

$$+ (\bar{c}_0 - \bar{c}_2)|\Phi_1|^2|\Phi_{-1}|^2 + \frac{\bar{c}_0}{2}|\Phi_0|^4 + \bar{c}_2(\Phi_1^*\Phi_{-1}^*\Phi_0^2 + \Phi_0^{*2}\Phi_1\Phi_{-1})\Big\}. \quad (2)$$

the effective 1D couplings $\bar{c}_{0,2}$ are represented by

$$\bar{c}_0 = c_0/2a_\perp^2, \quad \bar{c}_2 = c_2/2a_\perp^2, \quad (3)$$

where $a_\perp$ is the size of the transverse ground state. In this expression,

$$c_0 = \pi\hbar^2(a_0 + 2a_2)/3m, \qquad c_2 = \pi\hbar^2(a_2 - a_0)/3m, \quad (4)$$

where a_f – s-wave scattering lengths; m is the mass of the atom.

Special (integrable) choice for the coupling constants $\bar{c}_0 = \bar{c}_2 \equiv -c < 0$, equivalently scattering lengths $2a_0 = -a_2 > 0$. In the dimensionless form: $\Phi \to \{\Phi_1, \sqrt{2}\Phi_0, \Phi_{-1}\}^T$ the corresponding GPE take the form:

$$\begin{aligned}
&i\partial_t\Phi_1 + \partial_x^2\Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^*\Phi_0^2 = 0,\\
&i\partial_t\Phi_0 + \partial_x^2\Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^*\Phi_1\Phi_{-1} = 0, \quad (5)\\
&i\partial_t\Phi_{-1} + \partial_x^2\Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^*\Phi_0^2 = 0.
\end{aligned}$$

$F = 2$ hyperfine state is described by a normalized spinor wave vector

$$\Phi(x,t) = (\Phi_2(x,t), \Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t), \Phi_{-2}(x,t))^T, \tag{6}$$

whose components are labelled by the values of $m_F = 2, 1, 0, -1, -2$. Here the energy functional within mean-field theory is defined by

$$E_{\rm GP}[\Phi] = \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m} |\partial_x \Phi|^2 + \frac{\epsilon c_0}{2} n^2 + \frac{c_2}{2} \mathbf{f}^2 + \frac{\epsilon c_4}{2} |\Theta|^2 \right), \tag{7}$$

where $\epsilon = \pm 1$. The number density and the singlet-pair amplitude are defined by

$$n = (\vec{\Phi}, \vec{\Phi^*}) = \sum_{\alpha=-2}^{2} \Phi_\alpha \Phi_\alpha^*, \qquad \Theta = (\vec{\Phi}, s_0 \vec{\Phi}) = 2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2.$$

The coupling constants c_i are real and can be expressed in terms of the transverse confinement radius and the s-wave scattering lengths of atoms. Choosing $c_2 = 0$, $c_4 = 1$ and $c_0 = -2$ we obtain

$$i\partial_t \Phi_{\pm 2} + \partial_{xx} \Phi_{\pm 2} = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 2} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi_{\mp 2}^*,$$

$$i\partial_t \Phi_{\pm 1} + \partial_{xx}\Phi_{\pm 1} = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 1} - \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi^*_{\mp 1},$$
$$i\partial_t \Phi_0 + \partial_{xx}\Phi_0 = -2\epsilon(\vec{\Phi}, \vec{\Phi^*})\Phi_{\pm 0} + \epsilon(2\Phi_2\Phi_{-2} - 2\Phi_1\Phi_{-1} + \Phi_0^2)\Phi^*_0.$$

which is integrable by the inverse scattering method.

Lax pair is related to symmetric spaces Fordy, Kulish (1983) of **BD.I**-type:

$$\simeq \mathrm{SO(n+2)/SO(2)\times SO(n)}$$

with $n = 3$ and $n = 5$ respectively.

2 Symmetric and homogeneous spaces

Symmetric space: $\mathcal{M}$ is globally symmetric if each its point p is isolated invariant point under an involutive isometry:

$$\mathcal{K}(\mathcal{M}) = \mathcal{M}, \qquad \mathcal{K}^2 = 1\!\!1.$$

Cartan has classified all such involutions.

$\mathcal{M} \equiv \mathfrak{G}/\mathcal{H}$ where $\mathfrak{G}$ is simple and $\mathcal{H}$ is semisimple. Normally

$$\mathcal{H} \equiv \{K \in \mathfrak{G}, \quad \text{such that} \quad KJK^{-1} = J, \quad J \in \mathcal{H}\}.$$

Local coordinates:

$$Q(x) = [J, Q'(x)].$$

Typically

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} 0 & Q^{+}(x) \\ Q^{-}(x) & 0 \end{pmatrix},$$

But for BD.I-type symmetric spaces:

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix},$$

Effectively it is enough to properly specify $\mathfrak{G}$ and J in order to determine $\mathcal{M}$. The corresponding Lie algebra $\mathfrak{g}$ acquires $\mathbb{Z}_2$-grading:

$$\mathfrak{g} = \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)},$$

$$\mathfrak{g}^{(0)} \equiv \{X : X \in \mathfrak{g} \qquad \mathcal{K}(X) = X\}, \qquad \mathfrak{g}^{(1)} \equiv \{X : X \in \mathfrak{g} \qquad \mathcal{K}(Y) = -Y\},$$

The grading property:

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \in \mathfrak{g}^{(0)}, \qquad [\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \in \mathfrak{g}^{(0)}$$

The set of positive roots Δ^+ also splits into two subsets:

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+,$$

$$\Delta_0^+ \equiv \{\alpha : \quad \alpha(J) = 0\} \qquad \Delta_1^+ \equiv \{\alpha : \quad \alpha(J) = a > 0\}$$

3 Multicomponent nonlinear Schrödinger equations for BD.I. series of symmetric spaces

MNLS equations for the **BD.I.** series of symmetric spaces (algebras of the type $so(2r+1)$ and J dual to e_1) have the Lax representation $[L, M] = 0$ as follows

$$L\psi(x,t,\lambda) \equiv i\partial_x \psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \qquad (8)$$

$$M\psi(x,t,\lambda) \equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \qquad (9)$$

$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right] \qquad (10)$$

where

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1,0,\ldots 0,-1). \qquad (11)$$

The $2r-1$-vectors $\vec{q}$ and $\vec{p}$ have the form

$$\vec{q} = (q_2,\ldots,q_r,q_{r+1},q_{r+2},\ldots,q_{2r})^T, \qquad \vec{p} = (p_2,\ldots,p_r,p_{r+1},p_{r+2},\ldots,p_{2r})^T,$$

while the matrix s_0 represents the metric involved in the definition of $so(2r-1)$, therefore it is related to the metric S_0 associated with $so(2r+1)$ in the following manner

$$S_0 = \sum_{k=1}^{2r+1}(-1)^{k+1}E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (E_{kn})_{ij} = \delta_{ik}\delta_{nj} \qquad (12)$$

Next we will use

$$\vec{E}_1^{\pm} = (E_{\pm(e_1-e_2)}, \dots, E_{\pm(e_1-e_r)}, E_{\pm e_1}, E_{\pm(e_1+e_r)}, \dots, E_{\pm(e_1+e_2)}), \quad (13)$$

We will use also the "scalar product"

$$(\vec{q}\cdot\vec{E}_1^{+}) = \sum_{k=2}^{r} (q_k(x,t)E_{e_1-e_k} + q_{2r-k+2}(x,t)E_{e_1+e_k}) + q_{r+1}(x,t)E_{e_1}.$$

Then the generic form of the potentials $Q(x,t)$ related to these type of symmetric spaces is

$$Q(x,t) = (\vec{q}(x,t)\cdot\vec{E}_1^{+}) + (\vec{p}(x,t)\cdot\vec{E}_1^{-}), \quad (14)$$

where E_α are the Weyl generators of the corresponding Lie algebra and Δ_1^+ is the set of all positive roots of $so(2r+1)$ such that $(\alpha, e_1) = 1$. In fact $\Delta_1^+ = \{e_1, \quad e_1 \pm e_k, \quad k = 2, \dots, r\}$.

In terms of these notations the generic MNLS type equations connected to **BD.I**. acquire the form

$$\begin{aligned} &i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0, \\ &i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} + (\vec{p}, s_0\vec{p})s_0\vec{q} = 0, \end{aligned} \quad (15)$$

In the case of $r = 2$ if we impose the reduction $p_k = q_k^*$ and introduce the new variables $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$, $\Phi_{-1} = q_4$ then we reproduce the equations (119) with $F = 1$; if $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$, $\Phi_{-2} = q_6$ then we get the $F = 2$-case.

4 Inverse scattering method and reconstruction of potential from minimal scattering data

Herein we remind some basic features of the inverse scattering theory appropriate for the special case of $F = 2$ spinor BEC equations.

Solving the direct and the inverse scattering problem (ISP) for L uses the Jost solutions

$$\lim_{x\to -\infty} \phi(x,t,\lambda) e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x\to \infty} \psi(x,t,\lambda) e^{i\lambda J x} = \mathbb{1} \qquad (16)$$

and the scattering matrix $T(\lambda, t) \equiv \psi^{-1}\phi(x,t,\lambda)$. Due to the special choice of J and to the fact that the Jost solutions and the scattering matrix take values in the group $SO(2r+1)$ we can use the following block-matrix structure of $T(\lambda, t)$

$$T(\lambda, t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad (17)$$

where $\vec{b}^\pm(\lambda, t)$ and $\vec{B}^\pm(\lambda, t)$ are $2r-1$-component vectors, $\mathbf{T}_{22}(\lambda)$ is a $2r-1 \times 2r-1$ block and $m_1^\pm(\lambda)$, $c_1^\pm(\lambda)$ are scalar functions satisfying $c_1^+ = 1/2(\vec{b}^+ \cdot s_0\vec{b}^+)/m_1^+$, $c_1^- = 1/2(\vec{B}^- \cdot s_0\vec{B}^-)/m_1^-$.

The ISP is reduced to a Riemann-Hilbert problem (RHP) for the fundamental analytic solution (FAS) $\chi^\pm(x, t, \lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda, t)$

$$T(\lambda) = T_J^-(\lambda)D_J^+(\lambda)\hat{S}_J^+(\lambda) = T_J^+(\lambda)D_J^-(\lambda)\hat{S}_J^-(\lambda), \qquad (18)$$

Here $S_J^\pm$, $T_J^\pm$ upper- and lower-block-triangular matrices, while $D_J^\pm(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda, t)$ above. The explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda, t)$ is

$$S_J^{\pm}(t,\lambda) = \exp\left(\pm(\vec{\tau}^{\pm}(\lambda,t)\cdot\vec{E}_1^{\pm})\right), \qquad \tau^+ = \frac{b^-}{m_1^+}, \qquad \tau^- = \frac{B_1^+}{m_1^-} \tag{19}$$

$$T_J^{\pm}(t,\lambda) = \exp\left(\mp(\vec{\rho}^{\mp}(\lambda,t)\cdot\vec{E}_1^{\pm})\right), \qquad \rho^+ = \frac{b^+}{m_1^+}, \qquad \rho^- = \frac{B_1^-}{m_1^-},$$

$$D_J^+ = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \qquad D_J^- = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix}, \tag{20}$$

and

$$\mathbf{m}_2^+ = \mathbf{T}_{22} + \frac{\vec{b}^+\vec{b}^{-T}}{m_1^+}, \qquad \mathbf{m}_2^- = \mathbf{T}_{22} + \frac{s_0\vec{b}^-\vec{b}^{+T}s_0}{m_1^-}.$$

Then the FAS can be defined as:

$$\chi^{\pm}(x,t,\lambda) = \phi(x,t,\lambda)S_J^{\pm}(t,\lambda) = \psi(x,t,\lambda)T_J^{\mp}(t,\lambda)D_J^{\pm}(\lambda). \tag{21}$$

If $Q(x,t)$ evolves according to (119) then the scattering matrix and its elements satisfy the following linear evolution equations

$$\begin{aligned} & i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2 \vec{b}^{\pm}(t,\lambda) = 0, \qquad i\frac{d\vec{B}^{\pm}}{dt} \pm \lambda^2 \vec{B}^{\pm}(t,\lambda) = 0, \\ & i\frac{dm_1^{\pm}}{dt} = 0, \qquad i\frac{d\mathbf{m}_2^{\pm}}{dt} = 0, \end{aligned} \tag{22}$$

so $D^{\pm}(\lambda)$ can be considered as generating functionals of the integrals of motion.

The FAS for real λ are linearly related

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda) G_J(\lambda,t), \qquad G_{0,J}(\lambda,t) = S_J^-(\lambda,t) S_J^+(\lambda,t). \tag{23}$$

One can rewrite eq. (23) in an equivalent form for the FAS $\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,t,\lambda)e^{i\lambda Jx}$ which satisfy also the relation

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}. \tag{24}$$

Then these FAS satisfy

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda) G_J(x,\lambda,t), \qquad G_J(x,\lambda,t) = e^{-i\lambda Jx} G_{0,J}^-(\lambda,t) e^{i\lambda Jx}. \tag{25}$$

Obviously the sewing function $G_j(x,\lambda,t)$ is uniquely determined by the Gauss factors $S_J^{\pm}(\lambda,t)$. In view of eq. (19) we arrive to the following

Lemma 1. *Let the potential $Q(x,t)$ be such that the Lax operator L has no discrete eigenvalues. Then as minimal set of scattering data which determines uniquely the scattering matrix $T(\lambda,t)$ and the corresponding potential $Q(x,t)$ one can consider either one of the sets $\mathfrak{T}_i$, $i=1,2$*

$$\mathfrak{T}_1 \equiv \{\vec{\rho}^{+}(\lambda,t), \vec{\rho}^{-}(\lambda,t), \quad \lambda\in\mathbb{R}\}, \qquad \mathfrak{T}_2 \equiv \{\vec{\tau}^{+}(\lambda,t), \vec{\tau}^{-}(\lambda,t), \quad \lambda\in\mathbb{R}\}. \tag{26}$$

Obviously, given $\mathfrak{T}_i$ one uniquely recovers the sewing function $G_J(x,t,\lambda)$. In order to recover the corresponding scattering matrix $T(\lambda)$ one can use the fact that the RHP (25) with canonical normalization has unique regular solution. Then the generalized Gauss factors are recovered as limits:

$$S_J^{\pm}(\lambda) = \lim_{x\to-\infty} e^{i\lambda Jx}\xi^{\pm}(x,\lambda)e^{-i\lambda Jx}, \qquad T_j^{\mp}(\lambda)D_J^{\pm}(\lambda) = \lim_{x\to\infty} e^{i\lambda Jx}\xi^{\pm}(x,\lambda)e^{-i\lambda Jx}. \tag{27}$$

Given the solution $\xi^{\pm}(x,t,\lambda)$ one recovers $Q(x,t)$ via the formula

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm} J \widehat{\xi}^{\pm}(x,t,\lambda) \right). \qquad (28)$$

We impose also the standard reduction:

$$Q(x,t) = \epsilon Q^{\dagger}(x,t) \Leftrightarrow p_k = \epsilon q_k^{*}.$$

As a consequence we have

$$\vec{\rho}^{\,-}(\lambda,t) = \epsilon \vec{\rho}^{\,+,*}(\lambda,t), \qquad \vec{\tau}^{\,-}(\lambda,t) = \epsilon \vec{\tau}^{\,+,*}(\lambda,t).$$

5 Dressing method and soliton solutions

The soliton solutions can be constructed by Hirota method (Wadati, (2005)) and also by the dressing Zakharov-Shabat method (VSG et al, (2006).

The main goal of the Zakharov-Shabat dressing method: starting from a known solutions $\chi_0^{\pm}(x,t,\lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x,t)$

to construct new singular solutions $\chi_1^{\pm}(x,t,\lambda)$ of L with a potential $Q_{(1)}(x,t)$ with two additional singularities located at prescribed positions $\lambda_1^{\pm}$; the reduction $\vec{p} = \vec{q}^*$ ensures that $\lambda_1^- = (\lambda_1^+)^*$. It is related to the regular one by a dressing factor $u(x,t,\lambda)$

$$\chi_1^{\pm}(x,t,\lambda) = u(x,\lambda)\chi_0^{\pm}(x,t,\lambda)u_-^{-1}(\lambda). \qquad u_-(\lambda) = \lim_{x\to-\infty} u(x,\lambda) \quad (29)$$

Note that $u_-(\lambda)$ is a block-diagonal matrix. $u(x,\lambda)$ must satisfy

$$i\partial_x u + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x,\lambda)] = 0, \quad (30)$$

and the normalization condition $\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}$.

The construction of $u(x,\lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ-dependence:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda)-1)P(x,t) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x,t), \qquad \overline{P} = S_0^{-1}P^T S_0, \quad (31)$$

where $P(x,t)$ and $\overline{P}(x,t)$ are projectors whose rank s can not exceed r and which satisfy $P\overline{P}(x,t) = 0$. Given a set of s linearly independent polarization vectors $|n_k\rangle$ spanning the corresponding eigensubspase of L one can define

$$P(x,t) = \sum_{a,b=1}^{s} |n_a(x,t)\rangle M_{ab}^{-1} \langle n_b^\dagger(x,t)|, \qquad M_{ab}(x,t) = \langle n_b^\dagger(x,t)|n_a(x,t)\rangle,$$

$$|n_a(x,t)\rangle = \chi_0^+(x,t,\lambda^+)|n_{0,a}\rangle, \quad c(\lambda) = \frac{\lambda-\lambda^+}{\lambda-\lambda^-}, \quad \langle n_{0,a}|S_0|n_{0,b}\rangle = 0. \tag{32}$$

Taking the limit $\lambda \to \infty$ in eq. (30) we get that

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^- - \lambda_1^+)[J, P(x,t) - \overline{P}(x,t)].$$

Below we list the explicit expressions only for the one-soliton solutions. To this end we assume $Q_{(0)} = 0$ and put $\lambda_1^\pm = \mu \pm i\nu$. As a result we get

$$q_k^{(1s)}(x,t) = -2i\nu \left(P_{1k}(x,t) + (-1)^k P_{\bar{k},2r+1}(x,t) \right), \tag{33}$$

where $\bar{k} = 2r+2-k$.

Repeating the above procedure N times we can obtain N soliton solutions.

5.1 The case of rank one solitons

In this case $s = 1$ so that the generic (arbitrary r) one-soliton solution reads

$$
\begin{aligned}
q_k &= \frac{-i\nu e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} \left(\alpha_k e^{z-i\phi_k} + (-1)^k \alpha_{\bar{k}} e^{-z+i\phi_{\bar{k}}}\right), \\
v &= \frac{\nu^2 - \mu^2}{\mu}, \qquad u = -2\mu, \qquad z(x,t) = \nu(x - ut - \xi_0), \\
\xi_0 &= \frac{1}{2\nu} \ln \frac{|n_{0,2r+1}|}{|n_{0,1}|}, \qquad \alpha_k = \frac{|n_{0,k}|}{\sqrt{|n_{0,1}||n_{0,2r+1}|}}, \qquad \Delta_0^2 = \frac{\sum_{k=2}^{2r} |n_{0,k}|^2}{2|n_{0,1} n_{0,2r+1}|},
\end{aligned} \tag{34}
$$

and $\delta_0 = \arg n_{0,1}/\mu = -\arg n_{0,2r+1}/\mu$, $\phi_k = \arg n_{0,k}$. The polarization vectors satisfy the following relation

$$
\sum_{k=1}^{r} 2(-1)^{k+1} n_{0,k} n_{0,\bar{k}} + (-1)^r n_{0,r+1}^2 = 0. \tag{35}
$$

Thus for $r=2$ we identify $\Phi_1 = q_2$, $\Phi_0 = q_3/\sqrt{2}$ and $\Phi_3 = q_4$ and we obtain the following solutions for the equation (119)

$$\begin{aligned}
\Phi_{\pm 1} &= -\frac{2i\nu\sqrt{\alpha_2\alpha_4}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 1}\cosh z_{\pm 1} - i\sin\phi_{\pm 1}\sinh z_{\pm 1}\right), \\
\delta_{\pm 1} &= \delta_0 \mp \frac{\phi_2 - \phi_4}{2\mu}, \qquad \phi_{\pm 1} = \frac{\phi_2 + \phi_4}{2} \qquad z_{\pm 1} = z \mp \frac{1}{2}\ln\frac{\alpha_4}{\alpha_2}, \\
\Phi_0 &= -\frac{\sqrt{2}i\nu\alpha_3 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_3\sinh z - i\sin\phi_3\cosh z\right).
\end{aligned}$$

For $r=3$ we identify $\Phi_2 = q_2$, $\Phi_1 = q_3$, $\Phi_0 = q_4$, $\Phi_{-1} = q_5$ and $\Phi_{-2} = q_6$, so that the one-soliton solution for equation (**??**) reads

$$\begin{aligned}
\Phi_{\pm 2} &= -\frac{2i\nu\sqrt{\alpha_2\alpha_6}e^{-i\mu(x-vt-\delta_{\pm 2})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 2}\cosh z_{\pm 2} - i\sin\phi_{\pm 2}\sinh z_{\pm 2}\right), \\
\Phi_{\pm 1} &= -\frac{2i\nu\sqrt{\alpha_3\alpha_5}e^{-i\mu(x-vt-\delta_{\pm 1})}}{\cosh 2z + \Delta_0^2}\left(\cos\phi_{\pm 1}\sinh z_{\pm 1} - i\sin\phi_{\pm 1}\cosh z_{\pm 1}\right), \\
\delta_{\pm 2} &= \delta_0 \mp \frac{\phi_2 - \phi_6}{2\mu}, \qquad \phi_{\pm 2} = \frac{\phi_2 + \phi_6}{2} \qquad z_{\pm 2} = z \mp \frac{1}{2}\ln\frac{\alpha_6}{\alpha_2},
\end{aligned}$$

$$\delta_{\pm 1} = \delta_0 \mp \frac{\phi_3 - \phi_5}{2\mu}, \qquad \phi_{\pm 1} = \frac{\phi_3 + \phi_5}{2}, \qquad z_{\pm 1} = z \mp \frac{1}{2} \ln \frac{\alpha_5}{\alpha_3},$$

$$\Phi_0 = -\frac{2i\nu\alpha_4 e^{-i\mu(x-vt-\delta_0)}}{\cosh 2z + \Delta_0^2} \left(\cos\phi_4 \cosh z - i \sin\phi_4 \sinh z\right).$$

Choosing appropriately the polarization vectors $|n\rangle$ we are able to reproduce the soliton solutions obtained by Wadati et al. both for $F = 1$ and $F = 2$ BEC.

6 Effects of reductions on soliton solutions

The reduction group G_R (Mikhailov, 1978) is a finite group which preserves the Lax representation so that the reduction constraints are automatically compatible with the evolution.

G_R must have two realizations:

i) $G_R \subset \mathrm{Aut}\mathfrak{g}$ and

ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair:

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J, \qquad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I, \qquad (36)$$

of the Lax representation:

$$\begin{array}{lll} 1) & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\ 2) & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\ 3) & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\ 4) & C_4(U(\kappa_2(\lambda))) = U(\lambda), & C_4(V(\kappa_2(\lambda))) = V(\lambda), \end{array}$$

6.1 N-wave system related to $so(5)$

Impose first a reductions of class 4 that does not affect the spectral parameter. Choose $C_2 = S_0$, $\kappa_2(\lambda) = \lambda$, so

$$S_0(U^T(\lambda))S_0^{-1} + U(\lambda) = 0, \qquad S_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Focus our attention on NLEE related to the $so(5)$ algebra. Thus the N-wave system itself consists of 8 equations. A half of them reads

$$\begin{aligned} i(J_1 - J_2)Q_{10,t}(x,t) - i(I_1 - I_2)Q_{10,x}(x,t) + kQ_{11}(x,t)Q_{\overline{01}}(x,t) = 0, \\ iJ_1Q_{11,t}(x,t) - iI_1Q_{11,x}(x,t) - k(Q_{10}Q_{01} + Q_{12}Q_{\overline{01}})(x,t) = 0, \\ i(J_1 + J_2)Q_{12,t}(x,t) - i(I_1 + I_2)Q_{12,x}(x,t) - kQ_{11}(x,t)Q_{01}(x,t) = 0, \\ iJ_2Q_{01,t}(x,t) - iI_2Q_{01,x}(x,t) + k(Q_{\overline{11}}Q_{12} + Q_{\overline{10}}Q_{11})(x,t) = 0. \end{aligned} \tag{37}$$

where $k := J_1I_2 - J_2I_1$ is a constant describing the wave interaction. The other 4 can be obtained by changing $Q_{kn} \leftrightarrow Q_{\overline{kn}}$. Dressing factor:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda) - 1)\,P(x) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x) \in SO(5), \tag{38}$$

$$\overline{P}(x) = S_0P^T(x)S_0^{-1}.$$

Generic 1-soliton solution reads

$$Q_{10}(z) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\left(e^{-i(\lambda^+ z_1 - \lambda^- z_2)} n_{0,1} m_{0,2} + e^{i(\lambda^+ z_2 - \lambda^- z_1)} n_{0,4} m_{0,5}\right),$$

$$Q_{11}(z) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\left(e^{-i\lambda^+ z_1} n_{0,1} m_{0,3} - e^{-i\lambda^- z_1} n_{0,3} m_{0,5}\right),$$

$$Q_{12}(z) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\left(e^{-i(\lambda^+ z_1 + \lambda^- z_2)} n_{0,1} m_{0,4} + e^{-i(\lambda^- z_1 + \lambda^+ z_2)} n_{0,2} m_{0,5}\right),$$

$$Q_{01}(z) = \frac{\lambda^- - \lambda^+}{\langle m|n\rangle}\left(e^{-i\lambda^+ z_2} n_{0,2} m_{0,3} + e^{-i\lambda^- z_2} n_{0,3} m_{0,4}\right),$$

$$\langle m|n\rangle = \sum_{k=1}^{5} e^{-i(\lambda^+ - \lambda^-) z_k} n_{0,k} m_{0,k}, \qquad z_k = J_k x + I_k t, \qquad k = 1, 2.$$

The other 4 field can be formally constructed by doing the following transformation

$$Q_{kn} \leftrightarrow Q_{\overline{kn}}, \qquad e^{-i\lambda^+ z_k} \leftrightarrow e^{i\lambda^- z_k}, \qquad n_{0,j} \leftrightarrow m_{0,j}.$$

A typical $\mathbb{Z}_2$ reduction: $KU^\dagger(\lambda^*)K^{-1} = U(\lambda)$ where $K = \operatorname{diag}(\epsilon_1, \epsilon_2, 1, \epsilon_2, \epsilon_1)$

with $\epsilon_k = \pm 1$.

$$J_k = J_k^*, \quad Q_{\overline{10}} = -\epsilon_1\epsilon_2 Q_{10}^*, \quad Q_{\overline{01}} = -\epsilon_2 Q_{01}^*, \quad Q_{\overline{11}} = -\epsilon_1 Q_{11}^*, \quad Q_{\overline{12}} = -\epsilon_1\epsilon_2 Q_{12}^*.$$

Reduced NLEE is given by 4 equation

$$\begin{aligned}
i(J_1 - J_2)Q_{10,t}(x,t) - i(I_1 - I_2)Q_{10,x}(x,t) - k\epsilon_2 Q_{11}(x,t)Q_{01}^*(x,t) = 0,\\
iJ_1 Q_{11,t}(x,t) - iI_1 Q_{11,x}(x,t) - k(Q_{10}Q_{01} + \epsilon_2 Q_{12}Q_{01}^*)(x,t) = 0,\\
i(J_1 + J_2)Q_{12,t}(x,t) - i(I_1 + I_2)Q_{12,x}(x,t) - kQ_{11}(x,t)Q_{01}(x,t) = 0,\\
iJ_2 Q_{01,t}(x,t) - iI_2 Q_{01,x}(x,t) - k\epsilon_1(Q_{11}^* Q_{12} + \epsilon_2 Q_{10}^* Q_{11})(x,t) = 0.
\end{aligned}$$

Then $\lambda^{\pm} = \mu \pm i\nu$, and $|m\rangle = K|n\rangle^*$ and 1-soliton solution becomes

$$Q_{10}(z) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(\epsilon_2 e^{-i(\lambda^+ z_1 - (\lambda^+)^* z_2)} n_{0,1} n_{0,2}^* + \epsilon_1 e^{i(\lambda^+ z_2 - (\lambda^+)^* z_1)} n_{0,4} n_{0,5}^* \right),$$

$$Q_{11}(z) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(e^{-i\lambda^+ z_1} n_{0,1} n_{0,3}^* - \epsilon_1 e^{-i(\lambda^+)^* z_1} n_{0,3} n_{0,5}^* \right),$$

$$Q_{12}(z) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(\epsilon_2 e^{-i(\lambda^+ z_1 + (\lambda^+)^* z_2)} n_{0,1} n_{0,4}^* + \epsilon_1 e^{-i((\lambda^+)^* z_1 + \lambda^+ z_2)} n_{0,2} n_{0,5}^* \right),$$

$$Q_{01}(z) = \frac{-2i\nu}{\langle n^*|K|n\rangle} \left(e^{-i\lambda^+ z_2} n_{0,2} n_{0,3}^* + \epsilon_2 e^{-i(\lambda^+)^* z_2} n_{0,3} n_{0,4}^* \right),$$

$$\langle n^*|K|n\rangle = \epsilon_1 |n_{0,1}|^2 e^{2\nu z_1} + \epsilon_2 |n_{0,2}|^2 e^{2\nu z_2} + |n_{0,3}|^2 + \epsilon_2 |n_{0,4}|^2 e^{-2\nu z_2} + \epsilon_1 |n_{0,5}|^2 e^{-2\nu z_1},$$

Solitons associated with subalgebras of $so(5)$:

1. Suppose $n_{0,1} = n_{0,5} = 0$. The only nonzero waves are $Q_{01}, Q_{\overline{01}}$ related to the simple root α_2 – a $so(3)$ soliton.

2. Another $sl(2)$ soliton is derived when $n_{0,2} = n_{0,4} = 0$. Then $Q_{11}, Q_{\overline{11}}$ are nonvanishing; the so(3) subalgebra is connected with the root $e_1 = \alpha_1 + \alpha_2$.

3. Let $n_{0,3} = 0$. Then $Q_{10}, Q_{\overline{10}}$ and $Q_{12}, Q_{\overline{12}}$ are nonzero waves. The corresponding subalgebra is $so(3) \oplus so(3) \approx so(4)$.

4. If $n^*_{0,1} = n_{0,5}$, $n^*_{0,2} = n_{0,4}$ and $n^*_{0,3} = n_{0,3}$ then

$$Q_{10}(z) = \frac{-i\nu}{\Delta_1} \sinh 2\theta_0 \cosh \nu(z_1 + z_2) e^{-i\mu(z_1 - z_2 - \delta_1 + \delta_2)},$$

$$Q_{11}(z) = -\frac{2\sqrt{2}i\nu}{\Delta_1} \sinh \theta_0 \sinh \nu z_1 e^{-i\mu(z_1 - \delta_1)},$$

$$Q_{12}(z) = \frac{-i\nu}{\Delta_1} \sinh 2\theta_0 \cosh \nu(z_1 - z_2) e^{-i\mu(z_1 + z_2 - \delta_1 - \delta_2)},$$

$$Q_{01}(z) = \frac{-2\sqrt{2}i\nu}{\Delta_1} \cosh \theta_0 \cosh \nu z_2 e^{-i\mu(z_2 - \delta_2)},$$

$$n_{0,1} = \frac{n_{0,3}}{\sqrt{2}} \sinh \theta_0 e^{i\mu\delta_1}, \qquad n_{0,2} = \frac{n_{0,3}}{\sqrt{2}} \cosh \theta_0 e^{i\mu\delta_2}, \qquad \theta_0 \in \mathbb{R},$$

$$\Delta_1(x,t) = 2\left(\sinh^2 \theta_0 \sinh^2(\nu z_1) + \cosh^2 \theta_0 \cosh^2(\nu z_2)\right).$$

If $\theta_0 = 0$ then a single wave remains nontrivial:

$$Q_{01}(x,t) = \frac{-\sqrt{2}i\nu}{\cosh \nu z_2} e^{-i\mu(z_2-\delta_2)}.$$

6.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions and Doublet Solitons

An additional $\mathbb{Z}_2$ symmetry:

$$\chi^-(x,\lambda) = K_1 \left((\chi^+)^\dagger(x,\lambda^*) \right)^{-1} K_1^{-1}$$

$$\chi^-(x,\lambda) = K_2 \left((\chi^+)^T(x,-\lambda) \right)^{-1} K_2^{-1}$$

where $K_{1,2} \in SO(5)$ and $[K_1, K_2] = 0$. Also $U(x,\lambda)$ satisfies both symmetry conditions. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-reduced 4-wave system reads

$$\begin{aligned}
(J_1 - J_2)\mathbf{q}_{10,t}(x,t) - (I_1 - I_2)\mathbf{q}_{10,x}(x,t) + k\mathbf{q}_{11}(x,t)\mathbf{q}_{01}(x,t) = 0,\\
J_1\mathbf{q}_{11,t}(x,t) - I_1\mathbf{q}_{11,x}(x,t) + k(\mathbf{q}_{12}(x,t) - \mathbf{q}_{10}(x,t))\mathbf{q}_{01}(x,t) = 0,\\
(J_1 + J_2)\mathbf{q}_{12,t}(x,t) - (I_1 + I_2)\mathbf{q}_{12,x}(x,t) - k\mathbf{q}_{11}(x,t)\mathbf{q}_{01}(x,t) = 0,\\
J_2\mathbf{q}_{01,t}(x,t) - I_2\mathbf{q}_{01,x}(x,t) + k(\mathbf{q}_{10}(x,t) + \mathbf{q}_{12}(x,t))q_{11}(x,t) = 0,
\end{aligned}$$

where $\mathbf{q}_{10}(x,t)$, $\mathbf{q}_{11}(x,t)$, $\mathbf{q}_{12}(x,t)$ and $\mathbf{q}_{01}(x,t)$ are real valued fields.

The dressing factor $u(x,\lambda)$ must be invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.

$$K_1 \left(u^\dagger(x,\lambda^*)\right)^{-1} K_1^{-1} = u(x,\lambda), \tag{39}$$

$$K_2 \left(u^T(x,-\lambda)\right)^{-1} K_2^{-1} = u(x,\lambda). \tag{40}$$

If $K_1 = K_2 = \mathbb{1}$ one way to satisfy both conditions is to choose the poles of $u(x,\lambda)$ at $\lambda^\pm = \pm i\nu$ and $|m(x,t)\rangle = |n(x,t)\rangle = e^{\nu(Jx+It)}|n_0\rangle$ real.

The **doublet solution** becomes

$$\mathbf{q}_{10}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 N_2 \cosh \nu[(J_1+J_2)x + (I_1+I_2)t - \xi_1 - \xi_2],$$
$$\mathbf{q}_{11}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 n_{0,3} \sinh \nu(J_1 x + I_1 t - \xi_1),$$
$$\mathbf{q}_{12}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_1 N_2 \cosh \nu[(J_1-J_2)x + (I_1-I_2)t - \xi_1 + \xi_2],$$
$$\mathbf{q}_{01}(x,t) = -\frac{4\nu}{\langle n|n\rangle} N_2 n_{0,3} \cosh \nu(J_2 x + I_2 t - \xi_2),$$
$$\langle n(x,t)|n(x,t)\rangle = 2N_1^2 \cosh 2\nu(J_1 x + I_1 t - \xi_1) + 2N_2^2 \cosh 2\nu(J_2 x + I_2 t - \xi_2) + n_{0,3}^2,$$

where

$$\xi_1 := \frac{1}{2\nu} \ln \frac{n_{0,5}}{n_{0,1}}, \qquad \xi_2 := \frac{1}{2\nu} \ln \frac{n_{0,4}}{n_{0,2}}, \qquad N_1 = \sqrt{n_{0,1} n_{0,5}}, \qquad N_2 = \sqrt{n_{0,2} n_{0,4}}.$$

6.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions and Quadruplet Solitons

Now the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariance of $u(x,t,\lambda)$ is ensured by adding two more terms:

$$u(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda - \lambda^+} + \frac{K_1 S A^*(x,t)(K_1 S)^{-1}}{\lambda - (\lambda^+)^*} - \frac{K_2 S A(x,t)(K_2 S)^{-1}}{\lambda + \lambda^+} - \frac{K_1 K_2 A^*(x,t)(K_1 K_2)^{-1}}{\lambda + (\lambda^+)^*}.$$

where $A(x,t) = |X(x,t)\rangle\langle F(x,t)|$ and

$$|F(x,t)\rangle = e^{i\lambda^+(Jx+It)}|F_0\rangle.$$

For $|X(x,t)\rangle$ we get a linear system of equations. Skipping the details we obtain the generic quadruplet solution to the 4-wave system associated with the $\mathbf{B}_2$ algebra

$$\mathbf{q}_{10} = \frac{4}{\Delta}\mathrm{Im}\left[a^* N_1 \cosh(\varphi_1 + \varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu \cosh(\varphi_1^* + \varphi_2) - i\nu \cosh(\varphi_1^* - \varphi_2)\right)\right] N_2$$

$$\mathbf{q}_{11} = \frac{4}{\Delta}\mathrm{Im}\,\left[a^* N_1 \sinh(\varphi_1) - \frac{im\lambda^+}{\mu\nu} N_1^* \sinh(\varphi_1^*)\right] m_0^3$$

$$\mathbf{q}_{12} = \frac{4}{\Delta}\mathrm{Im}\,\left[a^* N_1 \cosh(\varphi_1 - \varphi_2) - \frac{imN_1^*}{\mu\nu}\left(\mu\cosh(\varphi_1^* - \varphi_2) - i\nu\cosh(\varphi_1^* + \varphi_2)\right)\right] N_2$$

$$\mathbf{q}_{01} = \frac{4}{\Delta}\mathrm{Im}\,\left[a^* N_2 \cosh(\varphi_2) - \frac{im\lambda^{+*}}{\mu\nu} N_2^* \cosh(\varphi_2^*)\right] m_0^3.$$

where

$$a(x,t) = \frac{1}{\mu + i\nu}\left[N_1^2 \cosh 2\varphi_1 + N_2^2 \cosh 2\varphi_2 + \frac{F_{0,3}^2}{2}\right], \quad b(x,t) = \frac{m(x,t)}{i\nu},$$

$$c(x,t) = \frac{m(x,t)}{\mu}, \quad m(x,t) = |N_1|^2 \cosh(2\mathrm{Re}\,\varphi_1) + |N_2|^2 \cosh(2\mathrm{Re}\,\varphi_2) + \frac{|m_0^3|^2}{2},$$

$$N_\sigma := \sqrt{m_0^\sigma m_0^{6-\sigma}}, \quad \varphi_\sigma(x,t) := i\lambda^+(J_\sigma x + I_\sigma t) + \frac{1}{2}\log\frac{m_0^\sigma}{m_0^{6-\sigma}}, \quad \sigma = 1, 2.$$

Other **inequivalent** reductions: we can use automorphisms $\tilde{K}_1$ and/or $\tilde{K}_2$ taking values in the Weyl group.

7 The Generalized Fourier Transforms for Non-regular J

We show that the ISM can be viewed as generalized Fourier transform (GFT). We determine explicitly the proper generalizations of the usual exponents. We also introduce a skew–scalar product on $\mathcal{M}$ which provides it with a symplectic structure.

7.1 The Wronskian relations

Along with the Lax operator we consider associated systems:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,t,\lambda)U(x,t,\lambda) = 0, \qquad U(x,\lambda) = Q(x) - \lambda J, \tag{41}$$

$$i\frac{d\delta\psi}{dx} + \delta U(x,t,\lambda)\psi(x,t,\lambda) + U(x,t,\lambda)\delta\psi(x,t,\lambda) = 0 \tag{42}$$

$$i\frac{d\dot{\psi}}{dx} - \lambda J\psi(x,t,\lambda) + U(x,t,\lambda)\dot{\psi}(x,t,\lambda) = 0 \tag{43}$$

where $\delta\psi$ corresponds to a given variation $\delta Q(x,t)$ of the potential, while by dot we denote the derivative with respect to the spectral parameter.

We start with the identity:

$$(\hat{\chi} J\chi(x,\lambda) - J)|_{x=-\infty}^{\infty} = i\int_{-\infty}^{\infty} dx\, \hat{\chi}[J, Q(x)]\chi(x,\lambda), \tag{44}$$

where $\chi(x,\lambda)$ can be any fundamental solution of L.

One can use the asymptotics of $\chi^{\pm}(x,\lambda)$ for $x\to\pm\infty$to express the l.h.sides of the Wronskian relations in terms of the scattering data. Then

$$\begin{aligned} \langle \left(\hat{\chi}^{\pm} J\chi^{\pm}(x,\lambda) - J\right) E_\beta\rangle\Big|_{x=-\infty}^{\infty} &= i\int_{-\infty}^{\infty} dx\, \langle \Big([J, Q(x)] \boldsymbol{e}_\beta^{\pm}(x,\lambda)\Big)\rangle, \\ \langle \left(\hat{\chi}'^{,\pm} J\chi'^{,\pm}(x,\lambda) - J\right) E_\beta\rangle\Big|_{x=-\infty}^{\infty} &= i\int_{-\infty}^{\infty} dx\, \langle \Big([J, Q(x)] \boldsymbol{e}'^{,\pm}_\beta(x,\lambda)\Big)\rangle, \end{aligned} \tag{45}$$

where

$$
\begin{aligned}
e_\beta^{\pm}(x,\lambda) &= \chi^{\pm} E_\beta \hat{\chi}^{\pm}(x,\lambda), \qquad & \boldsymbol{e}_\beta^{\pm}(x,\lambda) &= P_{0J}(\chi^{\pm} E_\beta \hat{\chi}^{\pm}(x,\lambda)), \\
e_\beta^{\prime,\pm}(x,\lambda) &= \chi^{\prime,\pm} E_\beta \hat{\chi}^{\prime,\pm}(x,\lambda), \qquad & \boldsymbol{e}_\beta^{\prime,\pm}(x,\lambda) &= P_{0J}(\chi^{\prime,\pm} E_\beta \hat{\chi}^{\prime,\pm}(x,\lambda)),
\end{aligned}
\tag{46}
$$

are the natural generalization of the 'squared solutions' introduced first for the $sl(2)$-case. By P_{0J} we have denoted the projector $P_{0J} = \mathrm{ad}\,_J^{-1}\mathrm{ad}\,_J$ on the block-off-diagonal part of the corresponding matrix-valued function.

The right hand sides of eq. (46) can be written down with the skew–scalar product:

$$
[\![X, Y]\!] = \int_{-\infty}^{\infty} dx \langle X(x), [J, Y(x)]\rangle, \tag{47}
$$

where $\langle X, Y\rangle$ is the Killing form; in what follows we assume that the Cartan-Weyl generators satisfy $\langle E_\alpha, E_{-\beta}\rangle = \delta_{\alpha,\beta}$ and $\langle H_j, H_k\rangle = \delta_{jk}$. The product is skew-symmetric $[\![X, Y]\!] = -[\![Y, X]\!]$ and is non-degenerate on the space of allowed potentials $\mathcal{M}$. Thus we find

$$\begin{aligned} \rho_\beta^+ &= -i\,[\![Q(x), \boldsymbol{e}'^{,+}_\beta]\!], \qquad & \rho_\beta^- &= -i\,[\![Q(x), \boldsymbol{e}'^{,-}_{-\beta}]\!], \\ \tau_\beta^+ &= -i\,[\![Q(x), \boldsymbol{e}^+_{-\beta}]\!], \qquad & \tau_\beta^- &= -i\,[\![Q(x), \boldsymbol{e}^-_\beta]\!], \\ \vec{\rho}^{\,+} = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\rho}^{\,-} &= \frac{\vec{B}^-}{m_1^-}, \qquad & \vec{\tau}^{\,+} = \frac{\vec{b}^-}{m_1^+}, \qquad \vec{\tau}^{\,-} &= \frac{\vec{B}^+}{m_1^-}. \end{aligned} \tag{48}$$

Thus the mappings $\mathfrak{F} : Q(x,t) \to \mathfrak{T}_i$ can be viewed as generalized Fourier transform in which $\boldsymbol{e}^\pm_\beta(x,\lambda)$ and $\boldsymbol{e}'^{,\pm}_\beta(x,\lambda)$ can be viewed as generalizations of the standard exponentials.

We apply ideas similar to the ones above and get:

$$\begin{aligned} \delta\rho_\beta^+ &= -i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}'^{,+}_\beta]\!], \qquad & \delta\rho_\beta^- &= i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}'^{,-}_{-\beta}]\!], \\ \delta\tau_\beta^+ &= i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}^+_{-\beta}]\!], \qquad & \delta\tau_\beta^- &= -i\,[\![\mathrm{ad}_J^{-1}\delta Q(x), \boldsymbol{e}^-_\beta]\!], \end{aligned} \tag{49}$$

where $\beta \in \Delta_1^+$.

These relations are basic in the analysis of the related NLEE and their Hamiltonian structures. Assume that

$$\delta Q(x,t) = Q_t \delta t + \mathcal{O}((\delta t)^2). \tag{50}$$

Keeping only the first order terms with respect to δt we find:

$$\begin{aligned} \frac{d\rho_\beta^+}{dt} &= -i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}'^{,+}_\beta]\!], \qquad & \frac{d\rho_\beta^-}{dt} &= i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}'^{,-}_{-\beta}]\!], \\ \frac{d\tau_\beta^+}{dt} &= i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}^+_{-\beta}]\!], \qquad & \frac{d\tau_\beta^-}{dt} &= -i[\![\operatorname{ad}_J^{-1} Q_t(x), \boldsymbol{e}^-_\beta]\!], \end{aligned} \tag{51}$$

7.2 Completeness of the 'squared solutions'

Let us introduce the sets of 'squared solutions'

$$\{\boldsymbol{\Psi}\} = \{\boldsymbol{\Psi}\}_{\rm c} \cup \{\boldsymbol{\Psi}\}_{\rm d}, \qquad \{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}\}_{\rm c} \cup \{\boldsymbol{\Phi}\}_{\rm d}, \tag{52}$$

$$\begin{aligned} \{\boldsymbol{\Psi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^+_{-\alpha}(x,\lambda), \quad \boldsymbol{e}^-_\alpha(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\ \{\boldsymbol{\Psi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^\pm_{\mp\alpha;j}(x), \quad \dot{\boldsymbol{e}}^\pm_{\mp\alpha;j}(x), \quad \ddot{\boldsymbol{e}}^\pm_{\mp\alpha;j}(x), \quad \dddot{\boldsymbol{e}}^\pm_{\mp\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\}, \end{aligned} \tag{53}$$

$$\begin{aligned} \{\boldsymbol{\Phi}\}_{\rm c} &\equiv \left\{ \boldsymbol{e}^+_{\alpha}(x,\lambda), \quad \boldsymbol{e}^-_{-\alpha}(x,\lambda), \quad \lambda \in \mathbb{R}, \quad \alpha \in \Delta_1^+ \right\}, \\ \{\boldsymbol{\Phi}\}_{\rm d} &\equiv \left\{ \boldsymbol{e}^\pm_{\pm\alpha;j}(x), \quad \dot{\boldsymbol{e}}^\pm_{\pm\alpha;j}(x), \quad \ddot{\boldsymbol{e}}^\pm_{\pm\alpha;j}(x), \quad \dddot{\boldsymbol{e}}^\pm_{\pm\alpha;j}(x), \quad \alpha \in \Delta_1^+, \right\}, \end{aligned} \tag{54}$$

where $j = 1, \dots, N$ and the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L, the latter consisting of $2N$ discrete eigenvalues $\lambda_j^\pm \in \mathbb{C}_\pm$.

Theorem 1 (see V.S.G. (1998)). *The sets* $\{\boldsymbol{\Psi}\}$ *and* $\{\boldsymbol{\Phi}\}$ *form complete sets of functions in* $\mathcal{M}_J$. *The completeness relation has the form:*

$$\begin{aligned}\delta(x-y)\Pi_{0J} &= \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) \\ &\quad - 2i\sum_{j=1}^{N} (G_{1,j}^+(x,y) + G_{1,j}^-(x,y)),\end{aligned} \tag{55}$$

$$\begin{aligned}\Pi_{0J} &= \sum_{\alpha\in\Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha), \\ G_1^\pm(x,y,\lambda) &= \sum_{\alpha\in\Delta_1^+} \boldsymbol{e}_{\pm\alpha}^\pm(x,\lambda) \otimes \boldsymbol{e}_{\mp\alpha}^\pm(y,\lambda),\end{aligned} \tag{56}$$

$$G_{1,j}^\pm(x,y) = \sum_{\alpha\in\Delta_1^+} (\dot{\boldsymbol{e}}_{\pm\alpha;j}^\pm(x) \otimes \boldsymbol{e}_{\mp\alpha;j}^\pm(y) + \boldsymbol{e}_{\pm\alpha;j}^\pm(x) \otimes \dot{\boldsymbol{e}}_{\mp\alpha;j}^\pm(y). \tag{57}$$

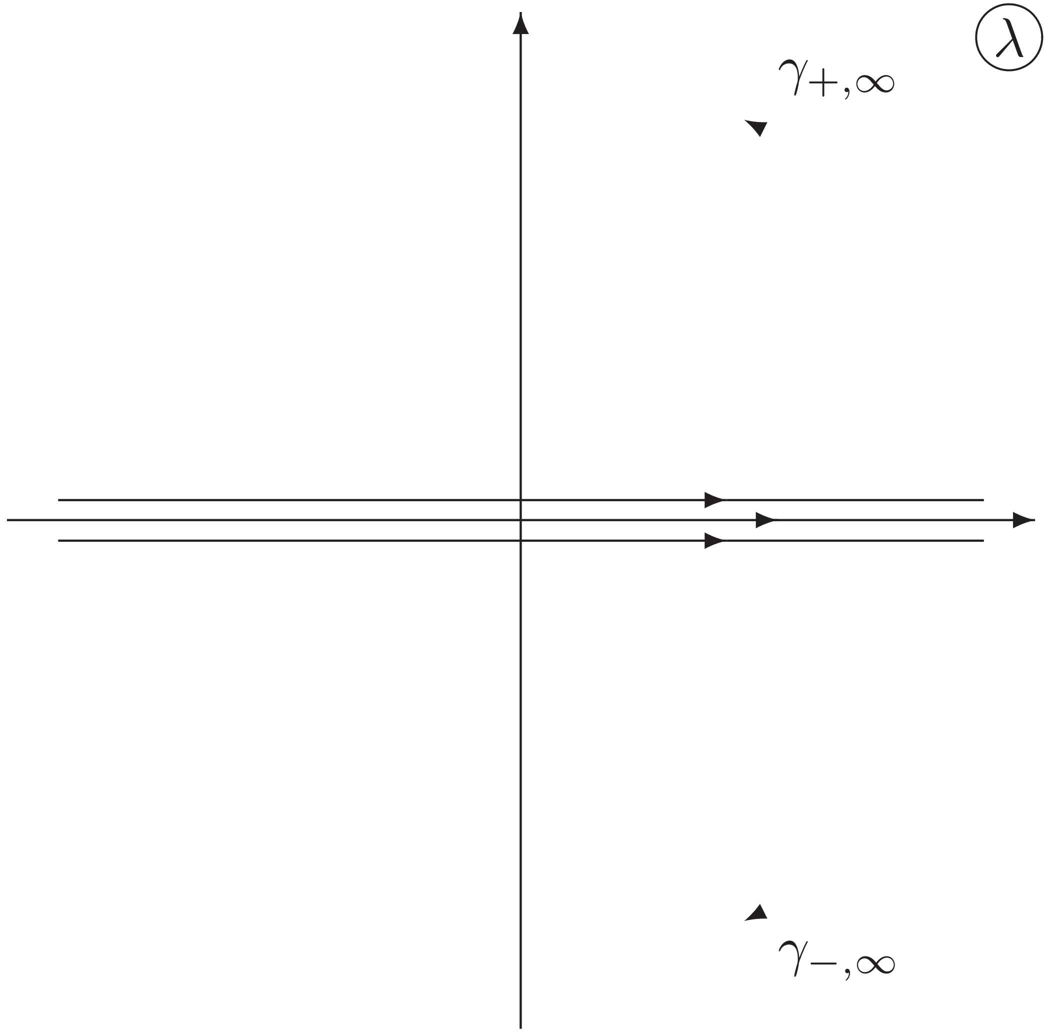
Idea of the proof. Apply the contour integration method to the function

$$
\begin{aligned}
G^{\pm}(x,y,\lambda) &= G_1^{\pm}(x,y,\lambda)\theta(y-x) - G_2^{\pm}(x,y,\lambda)\theta(x-y),\\
G_1^{\pm}(x,y,\lambda) &= \sum_{\alpha\in\Delta_1^+} \boldsymbol{e}_{\pm\alpha}^{\pm}(x,\lambda)\otimes \boldsymbol{e}_{\mp\alpha}^{\pm}(y,\lambda),\\
G_2^{\pm}(x,y,\lambda) &= \sum_{\alpha\in\Delta_0\cup\Delta_1^-} \boldsymbol{e}_{\pm\alpha}^{-}(x,\lambda)\otimes \boldsymbol{e}_{\mp\alpha}^{-}(y,\lambda) + \sum_{j=1}^{r} \boldsymbol{h}_j^{\pm}(x,\lambda)\otimes \boldsymbol{h}_j^{\pm}(y,\lambda),\\
\boldsymbol{h}_j^{\pm}(x,\lambda) &= \chi^{\pm}(x,\lambda)H_j\hat{\chi}^{\pm}(x,\lambda),
\end{aligned}
\tag{58}
$$

and calculate the integral

$$
\mathcal{J}_G(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda G^+(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda G^-(x,y,\lambda), \tag{59}
$$

in two ways: i) via the Cauchy residue theorem and ii) integrating along the contours. □



Фигура 1: The contours $\gamma_{\pm} = \mathbb{R} \cup \gamma_{\pm\infty}$.

Remark 1. *There is a dual completeness relation for the 'squared solutions' obtained by replacing all* $e^{\pm}_{\alpha}(x,\lambda)$ *with* $e'^{;\pm}_{\alpha}(x,\lambda)$.

7.3 Expansions over the „squared" solutions

Using the completeness relations one can expand any generic element $F(x)$ of the phase space $\mathcal{M}$ over each of the sets of 'squared solutions':

$$\begin{aligned} F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \Big(\boldsymbol{e}^{+}_{\alpha}(x,\lambda)\gamma^{+}_{F;-\alpha}(\lambda) - \boldsymbol{e}^{-}_{-\alpha}(x,\lambda)\gamma^{-}_{F;\alpha}(\lambda)\Big) \\ - 2i\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+} (Z^{+}_{F;\alpha,j}(x) + Z^{-}_{F;\alpha,j}(x)), \end{aligned} \tag{60}$$

$$F(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_1^+} \Big(\boldsymbol{e}_{-\alpha}^{+}(x,\lambda) \tilde{\gamma}_{F;\alpha}^{+}(\lambda) - \boldsymbol{e}_{\alpha}^{-}(x,\lambda) \tilde{\gamma}_{F;-\alpha}^{-}(\lambda) \Big)$$

$$+ 2i \sum_{j=1}^{N} \sum_{\alpha \in \Delta_1^+} (\tilde{Z}_{F;\alpha,j}^{+}(x) + \tilde{Z}_{F;\alpha,j}^{-}(x)), \qquad (61)$$

where

$$\gamma_{F;\alpha}^{\pm}(\lambda) = [\![\boldsymbol{e}_{\pm\alpha}^{\pm}(y,\lambda), F(y)]\!], \qquad \tilde{\gamma}_{F;\alpha}^{\pm}(\lambda) = [\![\boldsymbol{e}_{\mp\alpha}^{\pm}(y,\lambda), F(y)]\!], \qquad (62)$$

$$Z_{F;j}^{\pm}(x) = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \boldsymbol{e}_{\mp\alpha}^{\pm}(x,\lambda) \gamma_{F;\mp\alpha}^{\pm}(\lambda), \qquad \tilde{Z}_{F;j}^{\pm}(x) = \operatorname*{Res}_{\lambda=\lambda_j^{+}} \boldsymbol{e}_{\pm\alpha}^{\pm}(x,\lambda) \gamma_{F;\pm\alpha}^{+}(\lambda), \qquad (63)$$

Proposition 1. *The function $F(x) \equiv 0$ if and only if all its expansion coefficients vanish, i.e.:*

$$\gamma_{F;-\alpha}^{+}(\lambda) = \gamma_{F;\alpha}^{-}(\lambda) = 0, \qquad \alpha \in \Delta_1^+; \qquad Z_{F;\alpha,j}^{+}(x) = Z_{F;\alpha,j}^{-}(x) = 0;$$

$$\tilde{\gamma}_{F;\alpha}^{+}(\lambda) = \tilde{\gamma}_{F;-\alpha}^{-}(\lambda) = 0, \qquad \alpha \in \Delta_1^+; \qquad \tilde{Z}_{F;\alpha,j}^{+}(x) = \tilde{Z}_{F;\alpha,j}^{-}(x) = 0;$$

where $j = 1, \ldots, N$.

7.4 Expansions of $Q(x)$ and $\mathrm{ad}_J^{-1}\delta Q(x)$.

$$
\begin{aligned}
Q(x) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \big(\tau_\alpha^+(\lambda)\boldsymbol{e}_\alpha^+(x,\lambda) - \tau_\alpha^-(\lambda)\boldsymbol{e}_{-\alpha}^-(x,\lambda)\big) \\
&\quad - 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \tau_\alpha^+\boldsymbol{e}_\alpha^+(x,\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-} \tau_\alpha^-\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right),
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
Q(x) &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \big(\rho_\alpha^+(\lambda)\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \rho_\alpha^-(\lambda)\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\big) \\
&\quad + 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \rho_\alpha^+\boldsymbol{e}_\alpha^{\prime,+}(x,\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-} \rho_\alpha^-\boldsymbol{e}_\alpha^{\prime,-}(x,\lambda)\right),
\end{aligned}
\tag{65}
$$

$$\mathrm{ad}_J^{-1}\delta Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\tau_\alpha^+(\lambda)\boldsymbol{e}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right)$$
$$+ 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \delta\tau_\alpha^+\boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \delta\tau_\alpha^-\boldsymbol{e}_{-\alpha}^-(x,\lambda)\right), \tag{66}$$

$$\mathrm{ad}_J^{-1}\delta Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\rho_\alpha^+(\lambda)\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \delta\rho_\alpha^-(\lambda)\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda)\right)$$
$$- 2\sum_{j=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} \delta\rho_\alpha^+\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \delta\rho_\alpha^-\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda)\right). \tag{67}$$

These expansions combined with the proposition above give another way to establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$.

$$
\begin{aligned}
\mathrm{ad}\,_J^{-1}\frac{dQ}{dt} &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\frac{d\tau_\alpha^+}{dt}\boldsymbol{e}_\alpha^+(x,\lambda) + \frac{d\tau_\alpha^-}{dt}\boldsymbol{e}_{-\alpha}^-(x,\lambda) \right) \\
&\quad + 2\sum_{j=1}^N \sum_{\alpha\in\Delta_1^+} \left(\operatorname*{Res}_{\lambda=\lambda_j^+} \frac{d\tau_\alpha^+}{dt}\boldsymbol{e}_\alpha^+(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \frac{d\tau_\alpha^-}{dt}\boldsymbol{e}_{-\alpha}^-(x,\lambda) \right),
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
\mathrm{ad}\,_J^{-1}\frac{dQ}{dt} &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_{\alpha\in\Delta_1^+} \left(\frac{d\rho_\alpha^+}{dt}\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) + \frac{d\rho_\alpha^-}{dt}\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda) \right) \\
&\quad - 2\sum_{j=1}^N \sum_{\alpha\in\Delta_1^+} \left(\operatorname*{Res}_{\lambda=\lambda_j^+} \frac{d\rho_\alpha^+}{dt}\boldsymbol{e}_{-\alpha}^{\prime,+}(x,\lambda) - \operatorname*{Res}_{\lambda=\lambda_j^-} \frac{d\rho_\alpha^-}{dt}\boldsymbol{e}_{\alpha}^{\prime,-}(x,\lambda) \right).
\end{aligned} \tag{69}
$$

7.5 The generating operators

Introduce the generating operators $\Lambda_{\pm}$ through:

$$\begin{aligned} (\Lambda_+ - \lambda)\boldsymbol{e}^+_{-\alpha}(x,\lambda) &= 0, \qquad & (\Lambda_+ - \lambda)\boldsymbol{e}^-_{\alpha}(x,\lambda) &= 0, \\ (\Lambda_- - \lambda)\boldsymbol{e}^+_{\alpha}(x,\lambda) &= 0, \qquad & (\Lambda_- - \lambda)\boldsymbol{e}^-_{-\alpha}(x,\lambda) &= 0. \end{aligned} \tag{70}$$

Their derivation starts by introducing the splitting:

$$e^{\pm}_{\alpha}(x,\lambda) = e^{\mathrm{d},\pm}_{\alpha}(x,\lambda) + \boldsymbol{e}^{\pm}_{\alpha}(x,\lambda), \qquad e^{\mathrm{d},\pm}_{\alpha}(x,\lambda) = (\mathbb{1} - P_{0J})e^{\pm}_{\alpha}(x,\lambda), \tag{71}$$

into the equation

$$i\frac{de_{\alpha}}{dx} + [Q(x) - \lambda J, e_{\alpha}(x,\lambda)] = 0. \tag{72}$$

which is obviously satisfied by the 'squared solutions'. Then eq. (72) splits into:

$$i\frac{de^{\mathrm{d},\pm}_{\alpha}}{dx} + [Q(x), \boldsymbol{e}^{\pm}_{\alpha}(x,\lambda)] \;=\; 0, \tag{73}$$

$$i\frac{d\boldsymbol{e}_\alpha^{\pm}}{dx} + [Q(x), e_\alpha^{\mathrm{d},\pm}(x,\lambda)] = \lambda[J, \boldsymbol{e}_\alpha^{\pm}(x,\lambda)], \tag{74}$$

Eq. (73) can be integrated formally with the result

$$e_\alpha^{\mathrm{d},\pm}(x,\lambda) = C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda) + i\int_{\epsilon\infty}^{x} dy\,[Q(y), \boldsymbol{e}_\alpha^{\pm}(y,\lambda)], \tag{75}$$

$$C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda) = \lim_{y\to\epsilon\infty} e_\alpha^{\mathrm{d},\pm}(y,\lambda), \qquad \epsilon = \pm 1. \tag{76}$$

Next insert (75) into (74) and act on both sides by ad_J^{-1}. This gives us:

$$(\Lambda_\pm - \lambda)\boldsymbol{e}_\alpha^{\pm}(x,\lambda) = i[C_{\alpha;\epsilon}^{\mathrm{d},\pm}(\lambda), \mathrm{ad}_J^{-1} Q(x)], \tag{77}$$

where the generating operators $\Lambda_\pm$ are given by:

$$\Lambda_\pm X(x) \equiv \mathrm{ad}_J^{-1}\left(i\frac{dX}{dx} + i\left[Q(x), \int_{\pm\infty}^{x} dy\,[Q(y), X(y)]\right]\right). \tag{78}$$

$$(\Lambda_+ - \lambda)\boldsymbol{e}_{-\alpha}^{+}(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda)\boldsymbol{e}_{\alpha}^{-}(x,\lambda) = 0, \tag{79}$$

$$(\Lambda_- - \lambda)\boldsymbol{e}_{\alpha}^{+}(x,\lambda) = 0, \qquad (\Lambda_- - \lambda)\boldsymbol{e}_{-\alpha}^{-}(x,\lambda) = 0, \tag{80}$$

Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

8 Fundamental properties of the MNLS equations

8.1 The principal class of NLEE

By principle class of NLEE we mean the ones whose dispersion laws take the form:

$$F(\lambda) = f(\lambda)J, \tag{81}$$

where $f(\lambda)$ may be rational functions of λ whose poles lie outside the spectrum of L. The corresponding NLEE is

$$i\,\mathrm{ad}_J^{-1} Q_t + f(\Lambda_\pm)Q(x,t) = 0. \tag{82}$$

Theorem 2. *The NLEE* (82) *are equivalent to: i) the equations (22) and ii) to the following evolution equations for the generalized Gauss* *factors of* $T(\lambda)$*:*

$$i\frac{dS_J^+}{dt} + [F(\lambda), S_J^+] = 0, \qquad i\frac{dT_J^-}{dt} + [F(\lambda), T_J^-] = 0, \tag{83}$$

and

$$i\frac{dS_J^-}{dt} + [F(\lambda), S_J^-] = 0, \qquad i\frac{dT_J^+}{dt} + [F(\lambda), T_J^+] = 0. \tag{84}$$

8.2 The integrals of motion Hamiltonian properties of the MNLS eqs.

The block-diagonal Gauss factors $D_J^\pm(\lambda)$ are generating functionals of the integrals of motion. The principal series of integrals is generated by $m_1^\pm(\lambda)$:

$$\pm \ln m_1^\pm = \sum_{k=1}^{\infty} I_k \lambda^{-k}. \tag{85}$$

Let us outline a way to calculate their densities as functionals of $Q(x,t)$. Use a third type of Wronskian identities involving $\dot{\chi}^\pm(x,\lambda)$. They have the form:

$$\left(\hat{\chi}^{\pm}\dot{\chi}^{\pm}(x,\lambda) + iJx\right)\Big|_{x=-\infty}^{\infty} = -i\int_{-\infty}^{\infty} dx\ \left(\hat{\chi}J\chi(x,\lambda) - J\right), \tag{86}$$

which gives

$$\pm\frac{d}{d\lambda}\ln m_1^{\pm}(\lambda) = -i\int_{-\infty}^{\infty} dx\,(\langle\chi(x,\lambda)J\hat{\chi}J\rangle - 1). \tag{87}$$

Note that in the integrand of the above equation we have in fact $\langle h_1^{\pm}(x,\lambda)J\rangle$. Splitting $h_1^{\pm}(x,\lambda) = h_1^{d,\pm}(x,\lambda) + \boldsymbol{h}_1^{\pm}(x,\lambda)$ into 'block-diagonal' and 'block-off-diagonal' parts we get

$$\begin{aligned}(\Lambda_+ - \lambda)\boldsymbol{h}_1^{\pm}(x,\lambda) &= i\left[\lim_{y\to\pm\infty} h_1^{d,\pm}(x,\lambda), \operatorname{ad}_J^{-1}Q(x)\right] \\ &= i[J, \operatorname{ad}_J^{-1}Q(x)] \equiv Q(x),\end{aligned} \tag{88}$$

i.e.

$$\begin{aligned}(\Lambda_{\pm} - \lambda)\boldsymbol{h}_1^{\pm}(x,\lambda) &= Q(x), \\ h_1^{d,\pm}(x,\lambda) &= J + \int_{\pm\infty}^{x} dy\,[Q(y), \boldsymbol{h}_1^{\pm}(x,\lambda)].\end{aligned} \tag{89}$$

Using eq. (89) and inverting formally the operator $(\Lambda_\pm - \lambda)$ we obtain the relations:

$$\begin{aligned}
\pm\frac{d}{d\lambda}\ln m_1^\pm(\lambda) &= -i\int_{-\infty}^{\infty} dx\, \left(\left\langle J + \int_{\pm\infty}^{x} dy\, [Q(y), \boldsymbol{h}_1^\pm(x,\lambda)], J \right\rangle - 1 \right) \\
&= -i\int_{-\infty}^{\infty} dx\, \int_{\pm\infty}^{x} dy\, \left\langle [J, Q(y)], \boldsymbol{h}_1^\pm(x,\lambda) \right\rangle \\
&= -i\int_{-\infty}^{\infty} dx\, \int_{\pm\infty}^{x} dy\, \left\langle [J, Q(y)], (\Lambda_\pm - \lambda)^{-1} Q(x) \right\rangle .
\end{aligned} \tag{90}$$

This procedure allows us to express the integrals of motion as functionals of $Q(x)$ in compact form:

$$I_s = \frac{1}{s}\int_{-\infty}^{\infty} dx\, \int_{\pm\infty}^{x} dy\, \left\langle [J, Q(y)], \Lambda_\pm^s Q(x) \right\rangle . \tag{91}$$

Note: the operators Λ_+ and Λ_- produce the same integrals of motion.

Using the explicit form of $\Lambda_\pm$ we find that:

$$\begin{aligned}
\Lambda_\pm Q &= i\mathrm{ad}\,_J^{-1}\frac{dQ}{dx} = i\frac{dQ^+}{dx} - i\frac{dQ^-}{dx}, \\
\Lambda_\pm^2 Q &= -\frac{d^2Q}{dx^2} + \left[Q^+ - Q^-, [Q^+, Q^-]\right], \\
\Lambda_\pm^3 Q &= -i\frac{d^3Q^+}{dx^3} + i\frac{d^3Q^-}{dx^3} + 3i\left[Q^+, [Q_x^+, Q^-]\right] + 3i\left[Q^-, [Q^+, Q_x^-]\right],
\end{aligned} \tag{92}$$

$$Q^+(x,t) = (\vec{q}(x,t)\cdot\vec{E}_1^+), \qquad Q^-(x,t) = (\vec{p}(x,t)\cdot\vec{E}_1^-).$$

Thus for the first three integrals of motion we get:

$$\begin{aligned}
I_1 &= -i\int_{-\infty}^{\infty} dx\,\langle Q^+(x), Q^-(x)\rangle, \\
I_2 &= \frac{1}{2}\int_{-\infty}^{\infty} dx\,\left(\langle Q_x^+(x), Q^-(x)\rangle - \langle Q^+(x), Q_x^-(x)\rangle\right), \\
I_3 &= i\int_{-\infty}^{\infty} dx\,\left(-\langle Q_x^+(x), Q_x^-(x)\rangle + \frac{1}{2}\langle [Q^+(x), Q^-(x)], [Q^+(x), Q^-(x)]\rangle\right).
\end{aligned} \tag{93}$$

iI_1 – is the density of the particles, I_2 is the momentum and $-iI_3$ is the Hamiltonian of the MNLS equations. Indeed, taking $H_{(0)} = -iI_3$ with the Poissson brackets

$$\{q_k(y,t), p_j(x,t)\} = i\delta_{kj}\delta(x-y), \tag{94}$$

coincide with the MNLS equations (15). The above Poisson brackets are dual to the canonical symplectic form:

$$\begin{aligned}
\Omega_0 &= i\int_{-\infty}^{\infty} dx\, \mathrm{tr}\, \left(\delta\vec{p}(x) \wedge \delta\vec{q}(x)\right) \\
&= \frac{1}{i}\int_{-\infty}^{\infty} dx\, \mathrm{tr}\, \left(\mathrm{ad}_J^{-1}\delta Q(x) \wedge [J, \mathrm{ad}_J^{-1}\delta Q(x)\right) \qquad (95)\\
&= \frac{1}{i}[\![\mathrm{ad}_J^{-1}\delta Q(x) \underset{'}{\wedge} \mathrm{ad}_J^{-1}\delta Q(x)]\!], \qquad (96)
\end{aligned}$$

The last expression for Ω_0 is preferable to us because it makes obvious the interpretation of $\delta Q(x,t)$ as local coordinate on the co-adjoint orbit passing through J. It can be evaluated in terms of the scattering data variations.

$$\Omega_0 = \frac{1}{\pi i}\int_{-\infty}^{\infty} d\lambda\ \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda)\right) - 2\sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^+}\Omega_0^+(\lambda) + \operatorname*{Res}_{\lambda=\lambda_j^-}\Omega_0^-(\lambda)\right),$$

$$\Omega_0^\pm(\lambda) = \sum_{\alpha,\gamma\in\Delta_1^+}\delta\tau^\pm(\lambda)D^\pm_{\alpha,\gamma}\wedge\delta\rho^\pm_\gamma, \qquad D^\pm_{\alpha,\gamma} = \left\langle \hat{D}^\pm E_{\mp\gamma}D^\pm(\lambda)E_{\pm\alpha}\right\rangle,$$

Hierarchy of Hamiltonian formulations of MNLS:

$$\Omega_k = \frac{1}{i}[\![\mathrm{ad}_J^{-1}\delta Q \mathop{\wedge}_{,} \Lambda^k \mathrm{ad}_J^{-1}\delta Q]\!], \qquad \Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-), \qquad (97)$$

$$H_k = i^{k+3}I_{k+3}. \qquad (98)$$

We can also calculate Ω_k in terms of the scattering data variations. Doing this we will need also eqs. (79) and (80). The answer is

$$\Omega_k = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda\,\lambda^k\left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda)\right) - i\sum_{j=1}^{N}\left(\Omega^+_{k,j} + \Omega^-_{k;j}\right), \qquad (99)$$

$$\Omega_{k,j}^{\pm} = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \lambda^k \Omega_0^{\pm}(\lambda). \tag{100}$$

This allows one to prove that if we are able to cast Ω_0 in canonical form then all Ω_k will also be cast in canonical form and will be pair-wise equivalent.

II. Equations with Coxeter type reduction

This reduction is of the form:

$$4) \qquad C_4(U(\kappa_4(\lambda))) = U(\lambda), \qquad C_4(V(\kappa_4(\lambda))) = V(\lambda),$$

where C_4 is the Coxeter automorphism:

$$C_4^h = \mathbb{1}, \qquad \kappa_4(\lambda) = \omega\lambda, \qquad \omega^h = 1.$$

9 Recursion operator for generalized Zakharov-Shabat system with a $\mathbb{Z}_h$ Coxeter type reduction

Generalized Zakharov-Shabat system associated with a simple Lie algebra $\mathfrak{g}$ of rank r

$$L\psi = i\partial_x\psi + (q - \lambda J)\psi = 0, \tag{101}$$

where

$$q = \sum_{j=1}^{r} q_j H_j, \qquad J = \sum_{\alpha\in\mathcal{A}} E_\alpha. \tag{102}$$

The generators H_j for $j = 1, \ldots, r$ and E_α for any root $\alpha \in \Delta$ represent Cartan-Weyl's basis of the algebra $\mathfrak{g}$. The subset $\mathcal{A} \subset \Delta$ is formed by all admissible roots, so

$$\mathcal{A} = \{\alpha_1, \ldots, \alpha_r, \alpha_0\},$$

where α_0 is the minimal root of $\mathfrak{g}$

The above potential is obtained form a generic one by applying a $\mathbb{Z}_h$ reduction

$$\mathcal{C}q\mathcal{C}^{-1} = q, \qquad \mathcal{C}J\mathcal{C}^{-1} = \frac{1}{\omega}J, \tag{103}$$

where

$$\omega = e^{\frac{2\pi i}{h}}, \qquad \mathcal{C} = \exp\left(-\frac{2\pi i}{h} H_{\vec{\rho}_0}\right), \qquad (\vec{\rho}_0, \alpha_j) = 1, \qquad (104)$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots of $\mathfrak{g}$. Any root $\beta = \sum_{j=1}^{r} n_j \alpha_j$. Then

$$(\beta, \vec{\rho}_0) = \sum_{j=1}^{r} n_j = \mathrm{ht}\,(\beta),$$

i.e.

$$(\alpha_k, \vec{\rho}_0) = 1, \qquad (\alpha_0, \vec{\rho}_0) = h - 1.$$

Taking into account the famous formula

$$e^B A e^{-B} = e^{\mathrm{ad}\,_B} A$$

it follows

$$\mathcal{C} J \mathcal{C}^{-1} = \sum_{\alpha \in \mathcal{A}} \exp\left(-\frac{2\pi i}{h}\right) E_\alpha = \omega^{-1} J. \qquad (105)$$

Consider the algebra $\mathfrak{sl}(r+1)$. For $\mathfrak{sl}(r+1)$ we have

$$\mathcal{A} = \{e_i - e_{i+1}, \quad i = 1, \ldots, r; \quad e_{r+1} - e_1\}.$$

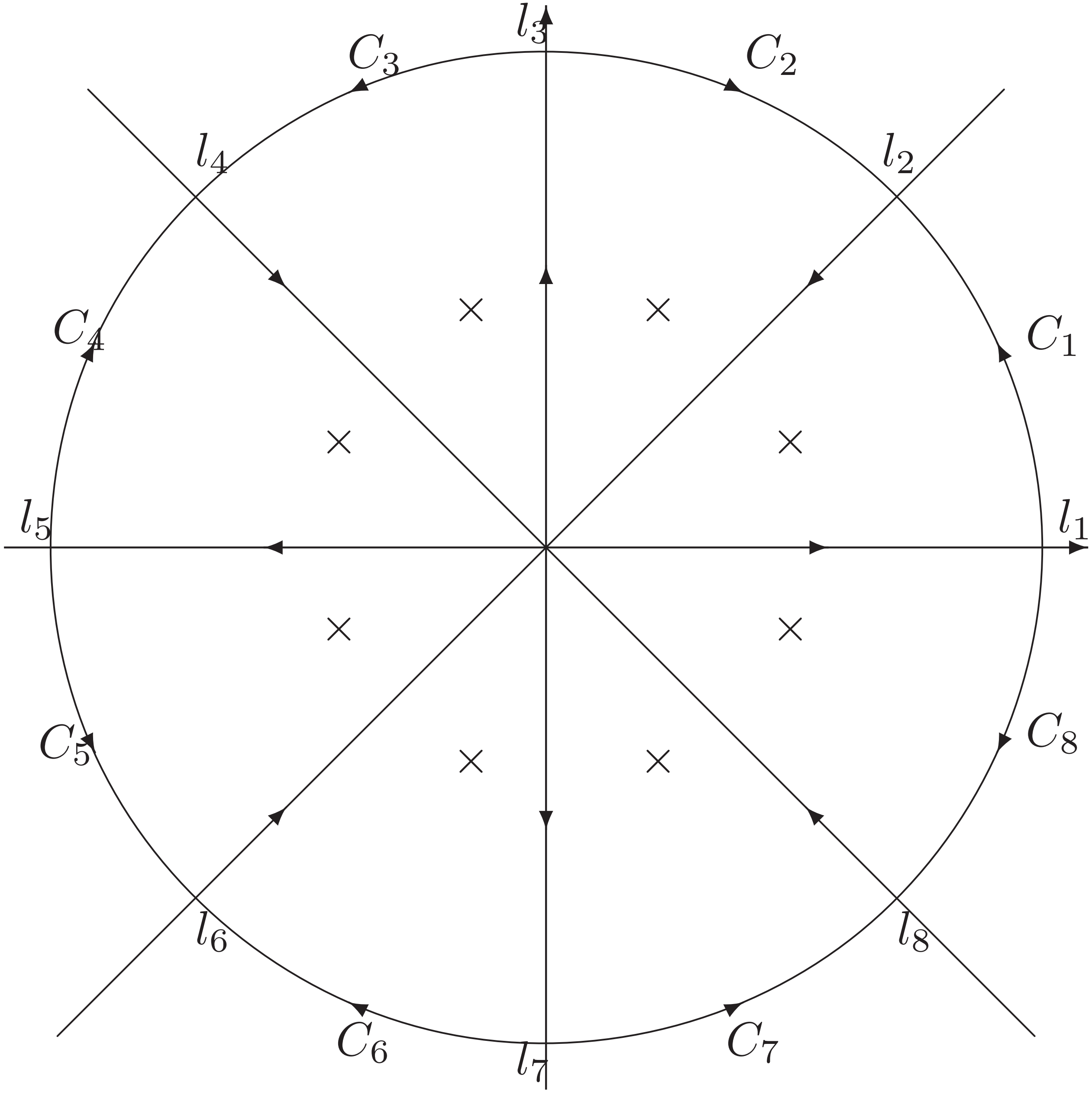
Choosing $\alpha = e_k - e_{k+1}$ we obtain $\vec{\rho}_0 = \sum_{j=1}^{r} \omega_j$. The minimal root is $\alpha = \alpha_{\min} = e_{r+1} - e_1$.

The Coxeter automorphism has a finite order $h = n$, the so-called Coxeter number. Hence it induces a $\mathbb{Z}_h$ grading in $\mathfrak{g}$ as follows

$$\mathfrak{g} = \sum_{k=0}^{h-1} \mathfrak{g}^k, \qquad \mathfrak{g}^k = \left\{X \in \mathfrak{g};\ \mathcal{C} X \mathcal{C}^{-1} = \omega^k J\right\}. \tag{106}$$

Comparing the reduction condition (103) with the definition of splitting of $\mathfrak{g}$ we see that

$$q \in \mathfrak{g}^0, \qquad J \in \mathfrak{g}^{h-1}. \tag{107}$$



l_3
C_3
C_2
l_4
l_2
C_4
C_1
l_5
l_1
C_5
C_8
l_6
l_8
C_6
l_7
C_7

The $\mathbb{Z}_h$ reduction affects the spectral properties of L — its continuous spectrum consists in $2h$ rays l_a $(a = 1, \dots, 2h)$ through the origin of coordinate system in the complex λ-plane. The angles between any adjacent rays are equal to π/h. The rays split into $2h$ sectors Ω_a. In each sector Ω_a there exists a fundamental analytic solution $\chi^a(x,\lambda)$. The fundamental analytic solutions of adjacent sectors are interrelated via a local Riemman-Hilbert problem

$$\chi^a(x,\lambda) = \chi^{a-1}(x,\lambda)G^a(\lambda). \tag{108}$$

Thus with each sector is associated "squared"solutions as follows

$$e^a_\alpha(x,\lambda) = \pi(\chi^a(x,\lambda)E_\alpha\hat{\chi^a}(x,\lambda)), \qquad h^a_j(x,\lambda) = \pi(\chi^a(x,\lambda)H_j\hat{\chi^a}(x,\lambda)), \tag{109}$$

where $\pi : \mathfrak{g} \mapsto \mathfrak{g}/\ker(\mathrm{ad}\,_J)$. Introducing

$$\mathcal{E}^a_\alpha = \chi^a E_\alpha \hat{\chi^a} = e^a_\alpha + d^a_\alpha, \qquad \mathcal{H}^a_j = \chi^a H_j \hat{\chi^a} = h^a_j + f^a_j. \tag{110}$$

we immediately convince ourselves that

$$i\partial_x \mathcal{E}^a_\alpha + [q - \lambda J, \mathcal{E}^a_\alpha] = 0, \tag{111}$$

$$i\partial_x \mathcal{H}_j^a + [q - \lambda J, \mathcal{H}_j^a] = 0. \tag{112}$$

Further on we shall skip the upper index a in the squared solutions for the sake of simplicity. After applying the splitting (110) to (111) we derive

$$\begin{aligned} i\partial_x e_\alpha + \pi[q, e_\alpha] + \pi[q, d_\alpha] &= \lambda\pi[J, e_\alpha], && (113)\\ i\partial_x d_\alpha + (\mathbb{1} - \pi)[q, e_\alpha] &= 0. && (114) \end{aligned}$$

Obviously, e_α and d_α possess the representation

$$\begin{aligned} e_\alpha(x,\lambda) &= \sum_{k=0}^{h-1} e_{\alpha,k}(x,\lambda), \qquad e_{\alpha,k}(x,\lambda) \in \mathfrak{g}^k,\\ d_\alpha(x,\lambda) &= \sum_{k=0}^{h-1} d_{\alpha,k}(x,\lambda), \qquad d_{\alpha,k}(x,\lambda) \in \mathfrak{g}^k. \end{aligned}$$

As a result we obtain the following equalities

$$i\partial_x e_{\alpha,0} + \pi[q, e_{\alpha,0}] = \lambda\pi[J, e_{\alpha,1}], \tag{115}$$

$$i\partial_x e_{\alpha,k} + \pi[q, e_{\alpha,k}] + \pi[q, d_{\alpha,k}] = \lambda\pi[J, e_{\alpha,k+1}], \qquad (116)$$

$k = 1, \dots, h-1$. Since d_α belongs to the centralizer C_J of J it is a linear combination of the following type

$$d_\alpha = \sum_{j=1}^{r} \mathbf{d}_\alpha^j \mathcal{E}_j, \qquad \mathcal{E}_j \in \mathfrak{g}^{k_j}, \quad [J, \mathcal{E}_j] = 0. \qquad (117)$$

Consider the $\mathfrak{sl}(r+1)$ case again ($h = r+1$). Now the adapted basis has the form

$$\mathcal{E}_k = J^{h-k} \in \mathfrak{g}^k.$$

It follows from (114) that

$$i\partial_x \mathbf{d}_\alpha^k + \frac{1}{h}\mathrm{tr}\,\left([q, e_\alpha]J^k\right) = 0, \qquad \Rightarrow\ \mathbf{d}_\alpha^k = \frac{i}{h}\int_{\pm\infty}^{x} dy\mathrm{tr}\,\left([q, e_\alpha]J^k\right)$$

On the other hand we have

$$i\partial_x e_{\alpha,0} + \pi[q, e_{\alpha,0}] = \lambda\pi[J, e_{\alpha,1}],$$
$$i\partial_x e_{\alpha,k} + \frac{i}{h}\pi[q, J^{h-k}] \int_{\pm\infty}^{x} dy \operatorname{tr}\left([q, e_{\alpha,k}]J^k\right) + \pi[q, e_{\alpha,k}] = \lambda\pi[J, e_{\alpha,k+1}].$$

As a result one obtains

$$e_{\alpha,1} = \frac{1}{\lambda}\Lambda_0 e_{\alpha,0}, \qquad e_{\alpha,k+1} = \frac{1}{\lambda}\Lambda_k e_{\alpha,k}, \quad k = 1, \ldots, h-1,$$

where

$$\Lambda_0 = \operatorname{ad}_J^{-1}\left(i\partial_x + \pi[q, .]\right),$$
$$\Lambda_k = \operatorname{ad}_J^{-1}\left(i\partial_x + \frac{i}{h}\pi\left([q, J^{h-k}]\right) \int_{\pm\infty}^{x} dy \operatorname{tr}\left([q, .]J^k\right) + \pi[q, .]\right).$$

Therefore

$$\Lambda e_{\alpha,0} = \lambda^h e_{\alpha,0}, \qquad \Lambda = \Lambda_{h-1}\Lambda_{h-2}\ldots\Lambda_0.$$

From Wronskian relations we get:

$$q(x) = \frac{i}{2\pi} \sum_{a=1}^{h} (-1)^{(a+1)} \beta_a(J) \int_{l_a} d\lambda \beta_a(J) \left(s^+_{a,\beta_a} e^{(a)}_{\beta_a,0}(x,\lambda) + s^-_{a,-\beta_a} e^{(a-1)}_{-\beta_a,0}(x,\lambda) \right),$$

$$\mathrm{ad}_J^{-1}[J^k, q(x)] = \frac{i}{2\pi} \sum_{a=1}^{h} (-1)^{(a+1)} \beta_a(J^k) \int_{l_a} d\lambda \beta_a(J) \left(s^+_{a,\beta_a} e^{(a)}_{\beta_a,0}(x,\lambda) + s^-_{a,-\beta_a} e^{(a-1)}_{-\beta_a,0}(x,\lambda) \right),$$

$$\Lambda^p \mathrm{ad}_J^{-1}[J^k, q(x)] = \frac{i}{2\pi} \sum_{a=1}^{h} (-1)^{(a+1)} \beta_a(J^k) \int_{l_a} d\lambda \lambda^{hp} \left(s^+_{a,\beta_a} e^{(a)}_{\beta_a,0}(x,\lambda) + s^-_{a,-\beta_a} e^{(a-1)}_{-\beta_a,0}(x,\lambda) \right),$$

and

$$\mathrm{ad}_J^{-1}\delta q(x) = \frac{i}{2\pi}\sum_{a=1}^{h}(-1)^a \int_{l_a} d\lambda \left(\delta s^+_{a,\beta_a} e^{(a)}_{\beta_a,h-1}(x,\lambda) - \delta s^-_{a,-\beta_a} e^{(a-1)}_{-\beta_a,h-1}(x,\lambda)\right).$$

If $\delta q(x) \simeq q(x,t+\delta t) - q(x,t) = q_t\delta t + \mathcal{O}((\delta t)^2)$, then

$$\mathrm{ad}_J^{-1} q_t(x) = \frac{i}{2\pi}\sum_{a=1}^{h}(-1)^a \int_{l_a} d\lambda \left(s^+_{a,\beta_a;t} e^{(a)}_{\beta_a,h-1}(x,\lambda) - s^-_{a,-\beta_a;t} e^{(a-1)}_{-\beta_a,h-1}(x,\lambda)\right).$$

Therefore the NLEE:

$$i\Lambda_{h-1}\mathrm{ad}_J^{-1} q_t + \sum_k c_k \Lambda_h\Lambda_{h-1}\dots\Lambda_k \mathrm{ad}_J^{-1}[J^k, q(x,t)] = 0,$$

is equivalent to the linear evolution eqs. for s^+_{a,β_a}:

$$i\frac{ds^+_{a,\beta_a}}{dt} \pm \sum_k c_k \lambda^{h-k+1}\beta_a(J^k) s^+_{a,\beta_a}(\lambda,t) = 0.$$

Examples of such NLEE:
The two-dimensional Toda field theory (Mikhailov, 1979):

$$\frac{\partial^2 u_k}{\partial x \partial t} = \exp(u_{k+1} - u_k) - \exp(u_k - u_{k-1}), \qquad k = 1, \dots, h \tag{118}$$

$u_0 \equiv u_h$;
$\mathbb{Z}_h$-NLS eq.:

$$iu_{k,t} + \gamma \left(\frac{\pi k}{N} \cdot u_{k,x} + i \sum_{p=1}^{N-1} u_p u_{k-p} \right)_x = 0, \qquad k = 1, 2, \dots, N-1, \tag{119}$$

Благодаря Thank you

за for

вниманието attention!