On low genus surfaces whose twistor lifts are harmonic sections

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0. Introduction.

$(\widetilde{M}, \widetilde{g})$: oriented 4-dim. Riemannian manifold

(M, g) : oriented surface

$f : M \to \widetilde{M}$: isometric immersion

$\Lambda^2_-(\widetilde{M})$: vector bundle of anti-selfdual 2-forms on $\widetilde{M}$

$f^\# \Lambda^2_-(\widetilde{M})$: pull-back bundle of $\Lambda^2_-(\widetilde{M})$ by f

$U(f^\# \Lambda^2_-(\widetilde{M}))$: unit sphere bundle of $f^\# \Lambda^2_-(\widetilde{M})$

$\widetilde{J} \in \Gamma(U(f^\# \Lambda^2_-(\widetilde{M})))$: twistor lift of M

$$\begin{array}{ccc} U(f^{\#}(\Lambda^2_-(\widetilde{M}))) \subset f^{\#}(\Lambda^2_-(\widetilde{M})) & \longrightarrow & \Lambda^2_-(\widetilde{M}) \\ \widetilde{J} \uparrow & & \downarrow \\ M & \xrightarrow[f]{} & \widetilde{M} \end{array}$$

• For the study of surfaces using the twistor lifts, see the following papers:

(1) E. Calabi (J. Diff. Geom., 1967),

(2) R. Bryant (J. Diff. Geom., 1982),

(3) T. Friedrich (Ann. Glob. Anal. Geom., 1984),

(4) I. Khemar (arXiv:math:DG/0803.3341v2) and $\cdots$

• **In this talk,**

surfaces whose twistor lifts are harmonic sections

are considered. In particular, we determine such surfaces in hyperKähler manifolds for low genus cases.

• **This talk is consists of**

1. **Twistor spaces and twistor lifts.**
2. **Harmonic sections.**
3. **Low genus cases.**
4. **Applications.**

1. Twistor spaces and twistor lifts.

$(\widetilde{M}, \widetilde{g})$: oriented 4-dim. Riemannian manifold

$\Lambda^2_-(\widetilde{M})$: vector bundle of anti-selfdual 2-forms on $\widetilde{M}$

(M, g) : oriented surface

$f : M \to \widetilde{M}$: isometric immersion

For each $x \in M$, take an orthonormal basis e_1, e_2, e_3, e_4 of $T_{f(x)}\widetilde{M}$ such that

$$\begin{cases} \text{(1) } e_1, e_2 \text{ are compatible with the orientation of } M, \\ \text{(2) } e_3, e_4 \text{ are normal to } T_xM, \\ \text{(3) } e_1, e_2, e_3, e_4 \text{ are compatible with the orientation of } \widetilde{M}. \end{cases}$$

$\omega_1, \omega_2, \omega_3, \omega_4$: dual basis of e_1, e_2, e_3, e_4.

Def. : The section $\widetilde{J} \in \Gamma(U(f^{\#}\Lambda^2_-(\widetilde{M})))$ defined by

$$\widetilde{J}(x) := \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4 \quad (x \in M)$$

is called the **twistor lift** of M.

• The unit sphere bundle $\mathcal{Z}(\widetilde{M}) := U(\Lambda^2_-(\widetilde{M}))$ is called the **twistor space** of $\widetilde{M}$.

• Using the metric $\widetilde{g}$, $\Lambda^2_-(\widetilde{M})$ can be identified with a subbundle Q of the bundle of all skew symmetric endomorphisms of $T\widetilde{M}$.

• $U(Q)(\cong U(\Lambda^2_-(\widetilde{M})))$ is the bundle whose fiber is consists of all complex structures preserving the the metric and orientation of $\widetilde{M}$.

• On the twistor space $\mathcal{Z}(\widetilde{M})$, an almost complex structure $J^{\mathcal{Z}}$ can be defined as follows :

K : connection map of $Q \cong \Lambda^2_-(\widetilde{M})$

(w.r.t. connection induced from the Levi-Civita connection of $\widetilde{M}$)

$p : \mathcal{Z}(\widetilde{M}) \to \widetilde{M}$: bundle projection

We have the decomposition

$$T_\phi \mathcal{Z}(\widetilde{M}) = T^h_\phi \mathcal{Z}(\widetilde{M}) \oplus T^v_\phi \mathcal{Z}(\widetilde{M})$$

where, $T^h_\phi \mathcal{Z}(\widetilde{M}) = \ker K_\phi$ and $T^v_\phi \mathcal{Z}(\widetilde{M}) = \ker p_{*\phi}$.

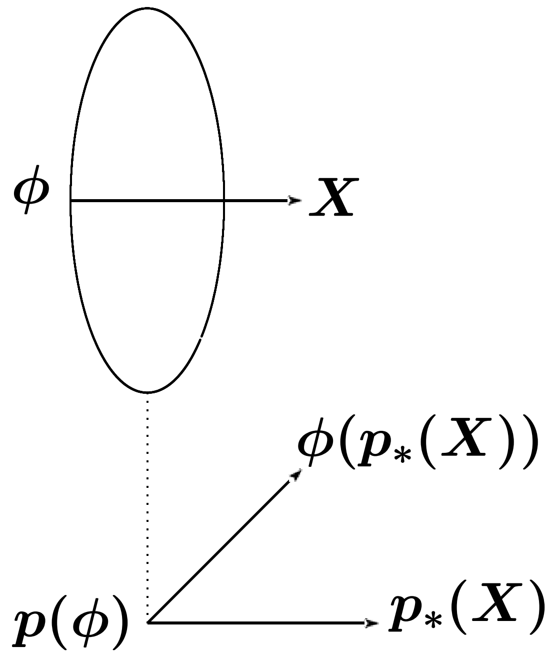
We define an almost complex structure $J^{\mathcal{Z}}$ by

$$J^{\mathcal{Z}}(X) = (\phi(p_*(X)))^h_\phi,$$

for $X \in T^h_\phi \mathcal{Z}(\widetilde{M})$, and

$$J^{\mathcal{Z}}(X) = \mathcal{J}(X)$$

for $X \in T^v_\phi \mathcal{Z}(\widetilde{M})$, where $\mathcal{J}$ is the canonical complex structure on each fiber ($\cong S^2$)



- **It is well-known that**

$$J^{\mathcal{Z}} \text{ is integrable} \iff \widetilde{M} \text{ is self-dual}$$

(M. F. Atiyah, N. J. Hitchin and I. M. Singer).

Def. : If the twistor lift $\widetilde{J}$ is horizontal map (that is, $\widetilde{\nabla}\widetilde{J} = 0$), the surface M is called **superminimal**.

- We define $J^{\perp}$ by

$$J^{\perp}(e_3) = -e_4 \text{ and } J^{\perp}(e_4) = e_3.$$

- Let h be the second fundamental form of M. Then we see that M is superminimal $\iff$ $h(JX, Y) = J^{\perp}h(X, Y)$ for all X, $Y \in TM$

Def. : If $(f_{\#} \circ \widetilde{J})_* \circ J = J^{\mathcal{Z}} \circ (f_{\#} \circ \widetilde{J})_*$, then the surface M is said to be **twistor holomorphic**.

- Define β by

$$\beta(X, Y) = h(X, JY) - J^{\perp}h(X, Y) + J^{\perp}h(JX, JY) + h(JX, Y)$$

for X, $Y \in TM$.

- M is twistor holomorphic $\iff$ $\beta = 0$.

- M is superminimal $\iff$ M is minimal and twistor holomorphic.

2. Harmonic section.

(M, g) : n-dim. compact Riemannian manifold

E : Riemannian vector bundle over M

g^E : fiber metric of E

∇^E : connection of E compatible with g^E

K^E : connection map of ∇^E

$p : E \to M$: bundle projection

We define the canonical metric G on E by

$$G(\zeta, \zeta) = g(p_*(\zeta), p_*(\zeta)) + g^E(K^E(\zeta), K^E(\zeta))$$

for all $\zeta \in TE$.

$U(E)$: unit sphere bundle of E

We give the induced metric of G on the submanifold $U(E)(\subset E)$.

$\mathcal{E}$: the energy functional on $C^\infty(M, U(E))$

Def. : The section $\xi \in \Gamma(U(E))$ is said to be harmonic section if it holds that

$$\frac{d}{dt}\mathcal{E}(\xi_t)\Big|_{t=0} = 0$$

for all variation $\xi_t \in \Gamma(U(E))$ of $\xi(= \xi_0)$.

- **In general, harmonic sections are not harmonic maps.**

- **The twistor lift** $\widetilde{J} \in \Gamma(U(f^{\#}\Lambda^2_-(\widetilde{M})))$ **is harmonic section** $\Longleftrightarrow$ $[\widetilde{J}, \bar{\triangle}^{\tilde{\nabla}}\widetilde{J}] = 0.$

3. Low genus cases.

M : compact surface

H : mean curvature vector field of M

$\nabla^{\perp}$: normal connection

We define $\delta\beta$ by

$$(\delta\beta)(X) = -\sum_{i=1}^{2} (\nabla'_{u_i}\beta)(u_i, X)$$

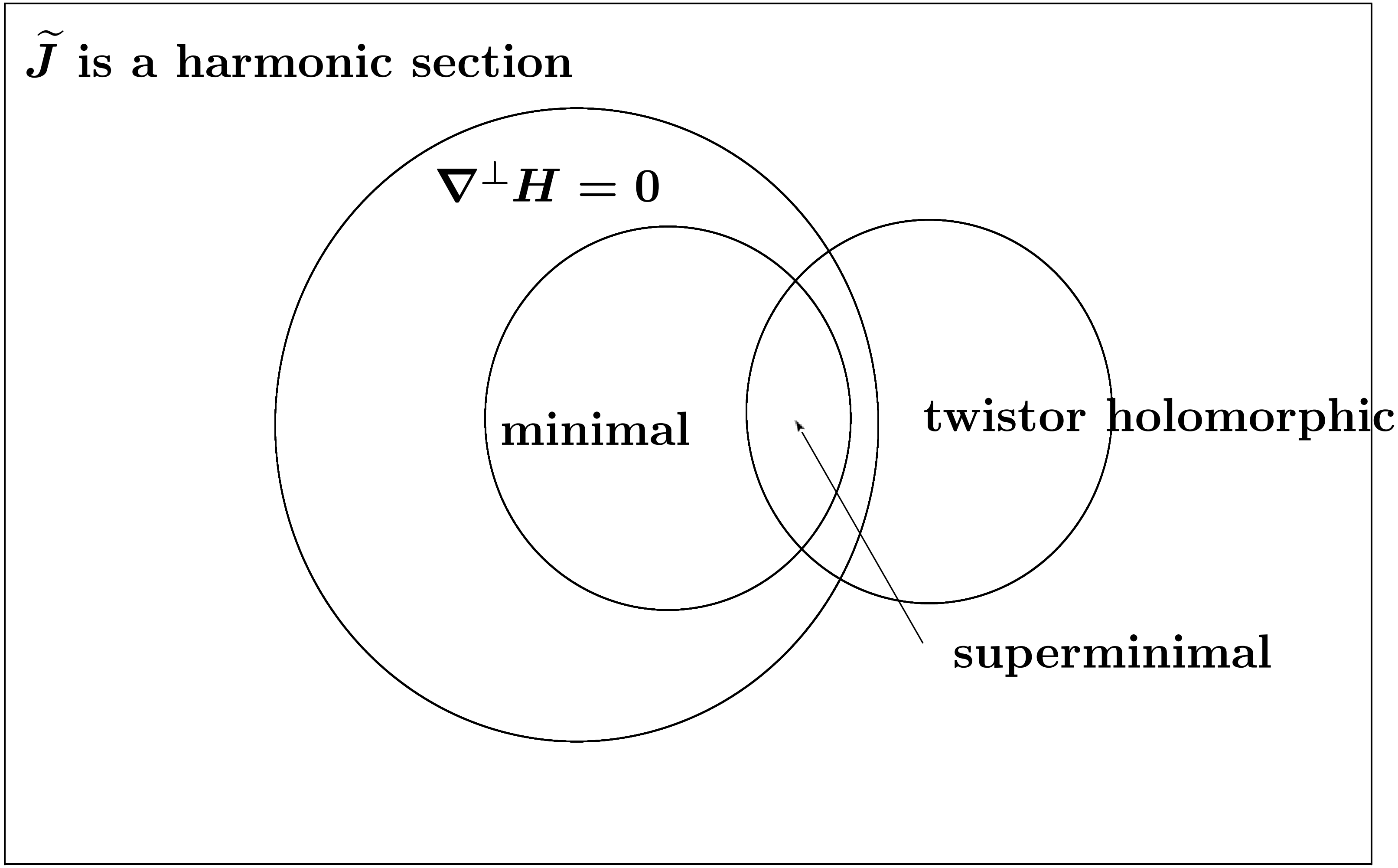
for all $X \in TM$, where u_1, u_2 is an orthonormal frame and $\nabla'\beta$ is the covariant derivative of β.

Thm. : If $\widetilde{M}$ is a self-dual Einstein manifold, then the following conditions are mutually equivariant :

(1) The twistor lift $\widetilde{J}$ of M is harmonic section.

(2) For all $X \in TM$, it holds that $\nabla^{\perp}_{JX} H = J^{\perp} \nabla^{\perp}_{X} H$.

(3) $\delta\beta = 0$.


$\widetilde{J}$ is a harmonic section
$\nabla^{\perp} H = 0$
minimal
twistor holomorphic
superminimal

$\widetilde{M}$: hyperkähler manifold

M : oriented, connected and compact surface in $\widetilde{M}$

$\chi(T^{\perp}M)$: Euler number of the normal bundle $T^{\perp}M$

q : genus of M

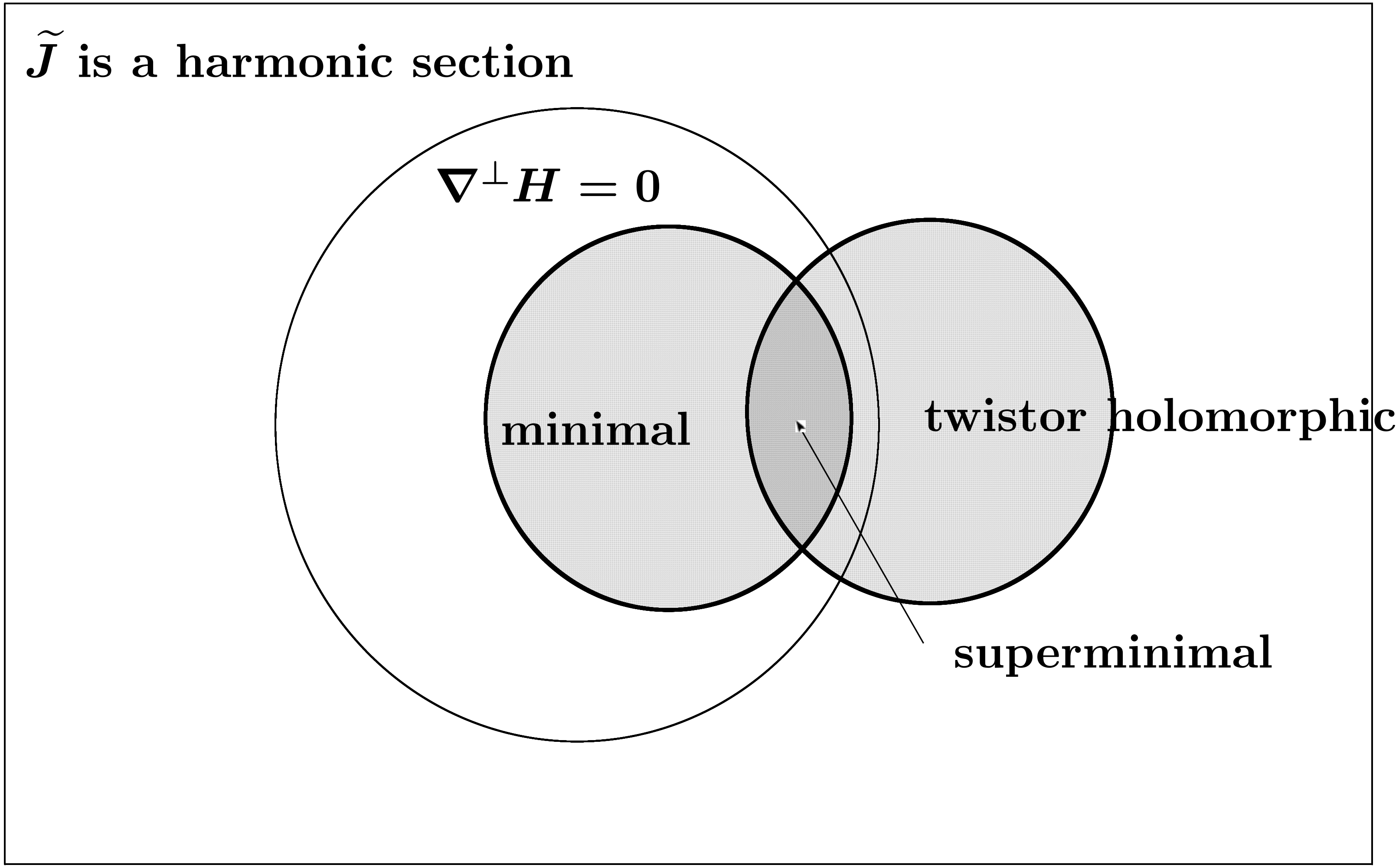
- **$\chi(T^{\perp}M) \in 2\mathbf{Z}$.**

Thm. : **Assume that the twistor lift** $\widetilde{J}$ **is a harmonic section and** $q = 0$**. Then we have**

(1) $\chi(T^{\perp}M) \geq 4 \Rightarrow M$ **is a non-superminimal minimal surface.**

(2) $\chi(T^{\perp}M) = 2 \Rightarrow M$ **is superminimal.**

(3) $\chi(T^{\perp}M) \leq 0 \Rightarrow M$ **is a non-superminimal twistor holomorphic surface.**



$\widetilde{J}$ is a harmonic section
$\nabla^{\perp} H = 0$
minimal
twistor holomorphic
superminimal

Similarly, we have

Thm : **Assume that the twistor lift $\widetilde{J}$ is a harmonic section and $q = 1$. Then we have**

(1) $\chi(T^{\perp}M) \geq 2 \Rightarrow M$ is a non-superminimal minimal surface.

(2) $\chi(T^{\perp}M) = 0 \Rightarrow \nabla^{\perp}H = 0$.

(3) $\chi(T^{\perp}M) \leq -2 \Rightarrow M$ is a non-superminimal twistor holomorphic surface.

- **There is a noncompact surface such that**

(1) $[\widetilde{J}, \bar{\triangle}^{\tilde{\nabla}}\widetilde{J}] = 0$ ($\widetilde{J}$ is a harmonic section),

(2) not twistor holomorphic,

(3) H is not parallel w.r.t. $\nabla^{\perp}$.

4. Applications.

When $\widetilde{M} = \mathrm{R}^4$, we have

Cor. : **Assume that M is an oriented, connected and compact surface in R^4. If the twistor lift of M is a harmonic section and $q = 0$, then M is twistor holomorphic.**

Cor. : **Assume that M is an oriented, connected and compact surface in R^4. If the twistor lift of M is a harmonic section and $q = 1$, then M is twistor holomorphic or CMC surface in R^3 or $S^3(r)$.**

Moreover, using this corollary, we also obtain the following results corresponding to "Hopf's Theorem" for a CMC surface in R^3.

Cor. (cf. D. Hoffman) : Assume that M is an oriented, connected and compact surface in R^4. If $\nabla^{\perp} H = 0$ and $q = 0$, then M is totally umbilic.