Ball Quotient Compactifications With a Co-Abelian Covering

Holzapfel's Conjecture On Ball Quotient Surfaces

- Conjecture: (Rolf-Peter Holzapfel - 1998) "... up to birational equivalence and compactifications, all complex algebraic surfaces are ball quotients".

- Let us consider the complex ball

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\} = SU_{2,1}/S(U_2 \times U_1)$$

and the ball lattices $\Gamma \subset SU_{2,1}$, i.e., the discrete subgroups with finite invariant measure of $\mathbb{B}/\Gamma$.

- Definition: A smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ of a ball quotient $\mathbb{B}/\Gamma$ is co-abelian if it has an abelian minimal model.

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Preliminary Results

- Holzapfel constructs:

- a smooth toroidal compactification $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$, whose abelian minimal model A_{-1} has decomposed complex multiplication by $\mathbb{Q}(i)$;

- a ball quotient compactification $\overline{\mathbb{B}/\Gamma_{K3,-1}^{(6,8)}}$, which is birational to the Kummer surface X_{-1} of A_{-1} and admits a double cover $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \to \overline{\mathbb{B}/\Gamma_{K3,-1}^{(6,8)}}$;

- a rational ball quotient compactification $\overline{\mathbb{B}/\Gamma_{rat,-1}^{(6,8)}}$ with $\mathbb{Z}[i]^* \times \mathbb{Z}[i]^*$-Galois cover $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \to \overline{\mathbb{B}/\Gamma_{rat,-1}^{(6,8)}}$.

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The Aim Of the Present Note

- Main Result - construction of ball quotient compactifications $\overline{\mathbb{B}/\Gamma}$, which are birational to hyperelliptic, Enriques or ruled surfaces with elliptic bases.

- All co-abelian smooth toroidal compactifications $(\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup T'$ with at most 3 rational (-1)-curves and minimal fundamental group of T' are Hirzebruch's $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)'$ and Holzapfel's $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)'$, $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$.

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- Let $(\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup \mathrm{T}'$ be a co-abelian smooth toroidal compactification, $\xi : (\mathbb{B}/\Gamma)' \to \mathrm{A}$ be the blow-down of the (-1)-curves to the abelian minimal model A and $\mathrm{T} = \xi(\mathrm{T}')$.

- Then $\mathrm{T} = \sum_{i=1}^{h} \mathrm{T}_i$ is a multi-elliptic divisor, i.e., T has smooth elliptic irreducible components T_i and the singular locus $\mathrm{T}^{sing} = \sum_{1 \leq i < j \leq h} (\mathrm{T}_i \cap \mathrm{T}_j)$ consists of their intersection points.

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- The group $G = \mathrm{Aut}(A, T)$ acts on the exceptional divisor of $\xi : (\mathbb{B}/\Gamma)' \to A$ and is isomorphic to $\mathrm{Aut}\left((\mathbb{B}/\Gamma)', T'\right)$.
- As a result, G acts on $\mathbb{B}/\Gamma$ and lifts to a ball lattice Γ_G, containing Γ as a normal subgroup with quotient $\Gamma_G/\Gamma = G$.
- Any subgroup H of G corresponds to a ball quotient compactification $\overline{\mathbb{B}/\Gamma_H}$, which is birational to A/H and admits an H-Galois covering $(\mathbb{B}/\Gamma)' \to \overline{\mathbb{B}/\Gamma_H}$.

Galois Quotients Of Co-Abelian Compactifications

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- The arithmetic lattice $\mathrm{SU}_{2,1}(\mathcal{O}_{-\mathrm{d}}) \subset \mathrm{SU}_{2,1}$ is called full Picard modular group over $\mathcal{O}_{-\mathrm{d}}$.

- If a ball lattice Γ is commensurable with $\mathrm{SU}_{2,1}(\mathcal{O}_{-\mathrm{d}})$, then Γ it is said to be a Picard modular group.

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Proposition: Let us suppose that the smooth toroidal compactification $(\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup \left(\sum_{i=1}^{h} T'_i \right)$ has abelian minimal model $A = E \times E$, contains s rational curves $\sum_{j=1}^{s} L_j$ with self-intersection (-1) and each smooth elliptic irreducible component T'_i intersects s_i among these L_j. If $s_1, \ldots, s_h$ take values $s'_1, \ldots, s'_t$ with multiplicities $k_1, \ldots, k_t$, $\sum_{i=1}^{t} k_i = h$, then the group $G = \mathrm{Aut}\left((\mathbb{B}/\Gamma)', \sum_{i=1}^{h} T'_i \right)$ is of cardinality

$$\mathrm{card}(G) \leq s\, k_1! \ldots k_t!\, \mathrm{card}(\mathrm{End}(E)^*).$$

Finite Automorphism Group Implies:

- Corollary 1: If $(\mathbb{B}/\Gamma)'$ is Picard modular co-abelian toroidal compactification then the ball lattice Γ_G with $\Gamma_G/\Gamma = G = \mathrm{Aut}\left((\mathbb{B}/\Gamma)', (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)\right)$ is also a Picard modular group.

- Corollary 2: The linear parts $g_o = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Gl}_2(\mathcal{O}_{-d})$ of all $g = \tau_{(U,V)} g_o \in \mathrm{Aut}(A, T)$ can be diagonalized.

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Diagonalizing Isogeny

- Thus, g_o with eigenvalues $\lambda_1 = \lambda_2$ are $g_o = \lambda_1 I_2$.
- If g_o has different eigenvalues $\lambda_1 \neq \lambda_2$ from $\mathcal{O}_{-d}$ then any isogeny $S \in \mathrm{Isog}(A) = \mathrm{Mat}_{2\times 2}(\mathcal{O}_{-d}) \cap \mathrm{Gl}_2(\mathbb{Q}(\sqrt{-d}))$ with

$$D_o = S^{-1} g_o S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

is called a diagonalizing isogeny for g_o.

- Let $S = \begin{pmatrix} \beta & \beta \\ \lambda_1 - \alpha & \lambda_2 - \alpha \end{pmatrix}$ for $\alpha \notin \{\lambda_1, \lambda_2\}$,
- $S = \begin{pmatrix} \lambda_1 - \lambda_2 & \beta \\ \gamma & \lambda_2 - \lambda_1 \end{pmatrix}$ for $\alpha = \lambda_1$, $\delta = \lambda_2$,
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- Let $\mathrm{h} = \tau_{(\mathrm{U},\mathrm{V})}\mathrm{h_o} \in \mathrm{Aut(A)}$ and $\mathrm{S} \in \mathrm{Isog(A)}$ be a diagonilizing isogeny of $\mathrm{h_o} \in \mathrm{Gl_2}(\mathcal{O}_{-\mathrm{d}})$.

- Then the Galois quotient $\mathrm{A}/\langle \mathrm{h} \rangle$ is a hyperelliptic surface if and only if

- (i) the eigenvalues of $\mathrm{h_o}$ are $\lambda_1 = 1$ and a primitive m-th root of unity $\lambda_2 \in \mathcal{O}^*_{-\mathrm{d}} \setminus \{1\}$, $\mathrm{m} > 1$,
- (ii) $(\mathrm{U},\mathrm{V}) \in \mathrm{A_{m-tor}}$,
- (iii) some (and therefore any) lifting $(\widetilde{\mathrm{U}},\widetilde{\mathrm{V}}) \in \mathbb{C}^2$ of $(\mathrm{U},\mathrm{V}) \in \mathrm{A}$ satisfies $\mathrm{S_{11}}\widetilde{\mathrm{V}} - \mathrm{S_{21}}\widetilde{\mathrm{U}} \in \mathrm{S_{11}}\mathcal{O}_{-\mathrm{d}} + \mathrm{S_{21}}\mathcal{O}_{-\mathrm{d}}$.

Hyperelliptic Quotient

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The Kummer Surface Of an Abelian Surface

- Any abelian surface A has automorphism $-I_2$ and $A/\langle -I_2 \rangle$ is a surface with 16 ordinary double points, covered by the 2-torsion points $A_{2-\mathrm{tor}}$ of A.

- The quotient $X = A_{\widehat{2-\mathrm{tor}}}/\langle -I_2 \rangle$ of the blow-up $A_{\widehat{2-\mathrm{tor}}}$ of A at $A_{2-\mathrm{tor}}$ is a smooth K3 surface, birational to $A/\langle -I_2 \rangle$ and called the Kummer surface of A.

$$\begin{array}{ccc} A_{\widehat{2-\mathrm{tor}}} & \longrightarrow & A \\ \langle -I_2 \rangle \downarrow & & \downarrow \langle -I_2 \rangle \\ X = A_{\widehat{2-\mathrm{tor}}}/\langle -I_2 \rangle & \longrightarrow & A/\langle -I_2 \rangle \end{array} .$$

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- Let X be the Kummer surface of an abelian surface A, $g = \tau_{(U,V)} g_o \in \mathrm{Aut}(A)$ and S be a diagonalizing isogeny of $g_o \in \mathrm{Gl}_2(\mathcal{O}_{-d})$.
- Then $Y = X/\langle g \rangle$ is an Enriques surface if and only if some (and therefore any) lifting $(\widetilde{U}, \widetilde{V}) \in \mathbb{C}^2$ of $(U, V) \in A$ satisfies the following conditions:

- (i) the eigenvalues of g_o are $\lambda_1 = 1$ and $\lambda_2 = -1$,
- (ii) $(U, V) \in A_{2-tor}$,
- (iii) $g_o(U, V) = (U, V)$,
- (iv) $S_{22}\widetilde{U} - S_{12}\widetilde{V} \not\in S_{22}\mathcal{O}_{-d} + S_{12}\mathcal{O}_{-d}$,
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- (i) the eigenvalues of g_o are $\lambda_1 = 1$ and $\lambda_2 = -1$,
- (ii) $(U, V) \in A_{2-tor}$,
- (iii) $g_o(U, V) = (U, V)$,
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Ruled Quotient With an Elliptic Or a Rational Base

- Proposition: (i) If a finite Galois quotient $S = A/H$ of an abelian surface A is a ruled surface then the base of S is of genus 1 or 0.

- (ii) If $g_o \in Gl_2(\mathcal{O}_{-d})$ is a linear automorphism of A then $X = A/\langle g_o \rangle$ is a ruled surface with an elliptic base if and only if the eigenvalues of g_o are $\lambda_1 = 1$ and $\lambda_2 \in \mathcal{O}^*_{-d} \setminus \{1\}$.

- (iii) If $g_o \in Gl_2(\mathcal{O}_{-d})$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 \in \mathcal{O}^*_{-d} \setminus \{1\}$ then for any $\lambda_3 \in \mathcal{O}^*_{-d} \setminus \{1\}$ the quotient $Y = A/\langle g_o, \lambda_3 I_2 \rangle$ is a ruled surface with a rational base and therefore a rational surface.

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Criterion For an Abelian Ball Quotient Model

- Theorem: (Holzapfel) The blow-up of an abelian surface A at the singular locus $T^{sing} = \sum_{1\leq i<j\leq h} (T_i \cap T_j)$ of a multi-elliptic divisor $T = \sum_{i=1}^{h} T_i$ is smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ of a ball quotient if and only if $A = E \times E$ and T has singularity rate

$$\frac{\sum_{i=1}^{h} \mathrm{card}(T_i \cap T^{sing})}{\mathrm{card}(T^{sing})} = 4.$$

- The smooth elliptic curves on $A = E \times E$ are of the form

$$E_{a_i,b_i} + (P_i, Q_i) = \{(a_i P + P_i, b_i P + Q_i) \,|\, P \in E\}.$$

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Holzapfel's Co-Abelian Compactification Over Gauss Numbers With 6 Exceptional Curves

Proposition: (Holzapfel - 2001) There is a smooth Picard modular $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$, such that the contraction of the rational (-1)-curves $\xi : \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \to A_{-1}$ provides the abelian surface $A_{-1} = E_{-1} \times E_{-1}$, $E_{-1} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ and the multi-elliptic divisor $\xi(T') = T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ with $T_k = E_{i^k,1}$ for $1 \leq k \leq 4$, $T_{m+4} = Q_m \times E_{-1}$, $T_{m+6} = E_{-1} \times Q_m$ for $1 \leq m \leq 2$, $Q_1 = \frac{1}{2}(\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}i)$, $Q_2 = iQ_1$.

- Proposition: The group $G_{-1}^{(6,8)} = \mathrm{Aut}(A_{-1}, T_{-1}^{(6,8)})$ is generated by the translation $\tau_{Q_{33}}$ with $Q_{33} = (Q_3, Q_3)$, $Q_3 = \frac{1+i}{2} (\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}i)$, the transposition θ of the elliptic factors of $A_{-1} = E_{-1} \times E_{-1}$ and the multiplications I, J by i on the first, respectively, the second factor of A_{-1}.

- The representation $\varphi : G_{-1}^{(6,8)} \to S_8(T_1, \ldots, T_8)$ has $\mathrm{Ker}\varphi = \langle \tau_{Q_{33}}(iI_2) \rangle \simeq \mathbb{Z}_4$ and $\mathrm{Im}\varphi$ of order 16, which is contained in $S_4(T_1, \ldots, T_4) \times S_4(T_5, \ldots, T_8)$ and surjects onto the dihedral groups $D_4(T_1, T_2, T_3, T_4)$ and $D_4(T_5, T_7, T_6, T_8)$.

The Automorphism Group Of $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$

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- Theorem: (i) The quotient of $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ by the cyclic group

$$\mathrm{H}_1 = \langle \tau_{\mathrm{Q}_{33}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \subset \mathrm{G}_{-1}^{(6,8)}$$

of order 2 is $\overline{\mathbb{B}/\Gamma_{\mathrm{HE},-1}^{(6,8)}}$ with hyperelliptic minimal model $\mathrm{A}_{-1}/\mathrm{H}_1$.

- (ii) The quotient of $\left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ by the subgroup

$$\mathrm{H}_2 = \langle -\mathrm{I}_2, \tau_{\mathrm{Q}_{33}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \subset \mathrm{G}_{-1}^{(6,8)}$$

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of order 4 is $\overline{\mathbb{B}/\Gamma_{rul,-1}^{(6,8)}}$, birational to a ruled surface with an elliptic base.

Hirzebruch's $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)'$ and Holzapfel's $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)'$, $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$ admit holomorphic involutions, leaving invariant their toroidal compactifying divisors. The corresponding orbit spaces are ball quotient compactifications $\overline{\mathbb{B}/\Gamma_{\mathrm{K3}}}$, birational to the Kummer surfaces $\mathrm{X}_{-\mathrm{d}}$ of the abelian minimal models $\mathrm{A}_{-\mathrm{d}}$.

Holzapfel's Co-Abelian Compactification Over Gauss Numbers With 3 Exceptional Curves

Proposition: (Holzapfel - 2001) There is a smooth Picard modular $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$, such that the contraction of the rational (-1)-curves $\xi : \left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)' \to A_{-1}$ yields the abelian surface $A_{-1} = E_{-1} \times E_{-1}$, $E_{-1} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ and the multi-elliptic divisor $\xi(T') = T_{-1}^{(3,6)} = \sum_{i=1}^{6} T_i$ with $T_1 = E_{1,0}$, $T_2 = E_{1,1+i}$, $T_3 = E_{1,1} + Q_{30}$, $T_4 = E_{1,i} + Q_{30}$, $T_5 = E_{1-i,1}$, $T_6 = E_{0,1}$, $Q_{30} = (Q_3, Q_0)$, $Q_3 = \frac{1+i}{2} (\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}i)$, $Q_0 = 0 (\mathrm{mod}\, \mathbb{Z} + \mathbb{Z}i)$.

The Automorphism Group And a Hyperelliptic Quotient Of $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$

- Proposition: The group $G_{-1}^{(3,6)} = \mathrm{Aut}(A_{-1}, T_{-1}^{(3,6)}) = \langle iI_2, \tau_{Q_{30}} \begin{pmatrix} -i & 1 \\ -i & 1+i \end{pmatrix}, \tau_{Q_{03}} \begin{pmatrix} 1 & 0 \\ 1 & i \end{pmatrix}, \tau_{Q_{03}} \begin{pmatrix} 1 & -1+i \\ 1 & -1 \end{pmatrix} \rangle$ is of order 96 and $\varphi : G_{-1}^{(3,6)} \to S_6(T_1, \ldots, T_6)$ has $\mathrm{Ker}\varphi = \langle iI_2 \rangle \simeq \mathbb{Z}_4$ and $\mathrm{Im}\varphi \simeq S_4$.

- Proposition: (i) The quotient of $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$ by the cyclic group

$$H_1 = \langle \tau_{Q_{30}} \begin{pmatrix} -i & 1 \\ 0 & 1 \end{pmatrix} \rangle \subset G_{-1}^{(3,6)}$$

is $\overline{\mathbb{B}/\Gamma_{HE,-1}^{(3,6)}}$ with hyperelliptic minimal model A_{-1}/H_1.

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- (ii) The involution $h_{-1}^{(3,6)} = \begin{pmatrix} 1 & 0 \\ 1+i & -1 \end{pmatrix} \in G_{-1}^{(3,6)}$ determines the ruled surface with an elliptic base $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'/\langle h_{-1}^{(3,6)} \rangle = \overline{\mathbb{B}/\Gamma_{rul,-1}^{(3,6)}}$ and the rational surface $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'/\langle h_{-1}^{(3,6)}, iI_2 \rangle = \overline{\mathbb{B}/\Gamma_{rat,-1}^{(3,6)}}$.

- (iii) The co-abelian smooth toroidal compactification $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$ is not a finite Galois cover of a ball quotient with Enriques minimal model.

Existence And Non-Existence Of $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'/H$

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Hirzebruch's Co-Abelian Compactification Over Eisenstein Numbers With 1 Exceptional Curve

- The ring of Eisenstein integers $\mathcal{O}_{-3} = \mathbb{Z} + \rho\mathbb{Z}$ with $\rho = e^{\frac{2\pi i}{6}}$ is the integers ring of $\mathbb{Q}(\sqrt{-3})$.

- Proposition: (Hirzebruch - 1984) There is a smooth Picard modular $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)'$, such that contraction of the rational (-1)-curves $\xi : \left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)' \to A_{-3}$ produces the abelian surface $A_{-3} = E_{-3} \times E_{-3}$, $E_{-3} = \mathbb{C}/\mathcal{O}_{-3}$ and the multi-elliptic divisor $\xi(T') = T_{-3}^{(1,4)} = \sum_{i=1}^{4} T_i$ with

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Proposition: The group

$$G_{-3}^{(1,4)} = \mathrm{Aut}(A_{-3}, T_{-3}^{(1,4)}) = \langle \rho I_2, \begin{pmatrix} 1 & 0 \\ 1 & -\rho \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \rangle$$

is of order 72 and $\varphi : G_{-3}^{(1,4)} \to S_4(T_1, \ldots, T_4)$ has $\mathrm{Ker}\varphi = \langle \rho I_2 \rangle \simeq \mathbb{Z}_6$ and $\mathrm{Im}\varphi = A_4$.

- Proposition: (i) The element $g_{-3}^{(1,4)} = \begin{pmatrix} 1 & 0 \\ 1 & -\rho \end{pmatrix} \in G_{-3}^{(1,4)}$ of order 3 determines a ruled surface with an elliptic base $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)' / \langle g_{-3}^{(1,4)} \rangle = \overline{\mathbb{B}/\Gamma_{rul,-3}^{(1,4)}}$ and a rational surface $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)' / \langle g_{-3}^{(1,4)}, -I_2 \rangle = \overline{\mathbb{B}/\Gamma_{rat,-3}^{(1,4)}}$.

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Proposition: The group
$\mathrm{G}_{-3}^{(3,6)} = \mathrm{Aut}(\mathrm{A}_{-3}, \mathrm{T}_{-3}^{(3,6)}) = \langle \tau_{\mathrm{P}_{01}}, \rho \mathrm{I}_2, \begin{pmatrix} 1 & -\rho\sqrt{-3} \\ 0 & -\rho \end{pmatrix} \rangle$ with
$\rho = \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}$, $\mathrm{P}_{01} = (\mathrm{P}_0, \mathrm{P}_1)$, $\mathrm{P}_0 = 0 (\mathrm{mod}\, \mathcal{O}_{-3})$, $\mathrm{P}_1 = \frac{1+\rho}{3} (\mathrm{mod}\, \mathcal{O}_{-3})$
is of order 54 and $\varphi : \mathrm{G}_{-3}^{(3,6)} \to \mathrm{S}_6(\mathrm{T}_1, \ldots, \mathrm{T}_6)$ has
$\mathrm{Ker}\varphi = \langle \rho^2 \mathrm{I}_2 \rangle \simeq \mathbb{Z}_3$ and $\mathrm{Im}\varphi = \mathrm{S}_3(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3) \times \mathrm{A}_3(\mathrm{T}_4, \mathrm{T}_5, \mathrm{T}_6)$.

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- (ii) The co-abelian smooth toroidal compactification $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)'$ does not admit finite Galois quotients, which are hyperelliptic or Enriques ball quotient compactifications.

- Proposition: (i) The element $g_{-3}^{(3,6)} = \begin{pmatrix} 1 & -\sqrt{-3} \\ 0 & \rho^2 \end{pmatrix} \in G_{-3}^{(3,6)}$ of order 3 determines the ruled surface with an elliptic base $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)' / \langle g_{-3}^{(3,6)} \rangle = \overline{\mathbb{B}/\Gamma_{rul,-3}^{(3,6)}}$ and the rational surface $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)' / \langle g_{-3}^{(3,6)}, \rho I_2 \rangle = \overline{\mathbb{B}/\Gamma_{rat,-3}^{(3,6)}}$.

- (ii) The co-abelian smooth toroidal compactification $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)'$ does not admit finite Galois quotients, which are hyperelliptic or Enriques ball quotient compactifications.

- Lemma: Let $\xi : (\mathbb{B}/\Gamma)' \to \mathrm{A} = \mathrm{E} \times \mathrm{E}$ be the blow-down of the (-1)-curves on a smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ and $\mathrm{T} = \xi(\mathrm{T}')$ be the image of the toroidal compactifying divisor $\mathrm{T}' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ on the abelian minimal model A.

- Then any smooth elliptic irreducible component $\mathrm{T_i}$ of T and its proper transform $\mathrm{T}'_{\mathrm{i}} \subset \mathrm{T}'$ admit a finite (not necessary Galois) covering $\mathrm{E} \to \mathrm{T_i} \simeq \mathrm{T}'_{\mathrm{i}}$.

Lower Bound On the Fundamental Groups

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Intersection Number

- Lemma: Let $A_{-d} = E_{-d} \times E_{-d}$ be an abelian surface with decomposed complex multiplication by $\mathbb{Q}(\sqrt{-d})$,
- $T_k = E_{a_k,b_k} + (P_k, Q_k)$ with $a_k, b_k \in \mathrm{End}(E_{-d})$, $k \in \{i,j\}$ be elliptic curves on A_{-d},
- $$\Lambda_k = a_k \pi_1(E_{-d}) + b_k \pi_1(E_{-d}) \subset \pi_1(E_{-d}).$$
- $$\Delta_{ij} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}, \quad N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} : \mathrm{End}(E_{-d}) \to \mathbb{Z}^{\geq 0},$$
- Then

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Multi-Elliptic And Toroidal Divisors With Minimal Fundamental Groups

- Definition: The irreducible components T'_{i} of T' or, equivalently, T_{i} of T have minimal fundamental groups if $\mathrm{T}'_{\mathrm{i}} \simeq \mathrm{T}_{\mathrm{i}} \simeq \mathrm{E}$ are isomorphic to the elliptic factor of the abelian minimal model $\mathrm{A} = \mathrm{E} \times \mathrm{E}$ of $(\mathbb{B}/\Gamma)'$.

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Uniqueness Result

Theorem: Up to an automorphism and a complex conjugation, Hirzebruch's $\left(\mathbb{B}/\Gamma_{-3}^{(1,4)}\right)'$ and Holzapfel's $\left(\mathbb{B}/\Gamma_{-3}^{(3,6)}\right)'$, $\left(\mathbb{B}/\Gamma_{-1}^{(3,6)}\right)'$ are the only co-abelian smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ with at most three rational (-1)-curves and minimal fundamental groups of $\mathrm{T}_{\mathrm{i}}' \subset \mathrm{T}' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$.

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- the existence of ball quotient compactifications,
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Towards The Ultimate Proof Of Holzapfel's Conjecture

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