A C-Spectral Sequence Associated with Free Boundary Variational Problems

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When a PDE is formalized as a natural geometrical object, one can use the common tools of differential calculus (e.g.: locality, differential cohomology, symmetries, etc.) to reveal some aspects of the PDE itself, **which could be hardly accessed by just using analytic techniques**.
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Introduction

Consider the problem of **finding a potential F of a closed 1–form ω in $\mathbb{R}^n$.**

When it is written **in coordinates**, it looks like a 1–st order PDE:

$$\frac{\partial F}{\partial x^i} = \omega_i \quad \omega = \omega_i dx^i. \tag{1}$$

Then elementary analysis allows to found a family of solutions $F + k$, where k is a real mumber arising from the process of integration.

From a **geometrical perspective**, our problem is in fact an aspect of the differential cohomology of $\mathbb{R}^n$.

If instead of the potentials of ω, we looked for the potentials of $i^*(\omega)$, where $i : \mathbb{R}^0 \subseteq \mathbb{R}^n$, then the solution would be straightforward: **all the real numbers** $k \in \mathbb{R}$. Now, $H(i^*)$ is an isomorphism, being $H(r^*)$ an its inverse, where $r : \mathbb{R}^n \to \mathbb{R}^0$ is the zero map.

So, if k is a potential of $i^*(\omega)$, then $r^*(k)$, i.e., k, must be a potential of $r^*(i^*(\omega))$, which differs from ω by the exact form $h(\omega)$, where h is the homotopy operator associated with the homotopy $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n}$.

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The potentials of ω are then $k - h(\omega)$.

We see that the true solution of our problem is k; the function $h(\omega)$ is just **an algebraic compensation due to the homotopy formula**. The problem in fact **lives on the 0–dimensional manifold** $\mathbb{R}^0$, where its solution is trivial, and, being formulated in terms of differential cohomology, it took advantage of the homotopy invariance of the latter. On the other hand, the analitic machinery drew attention away from the more conceptual, and hence more interesting, aspects of the problem.

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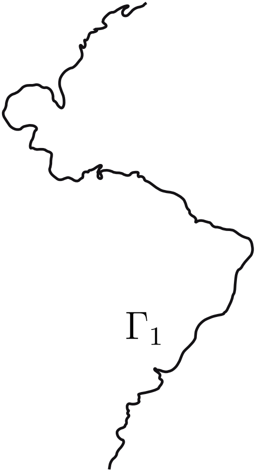
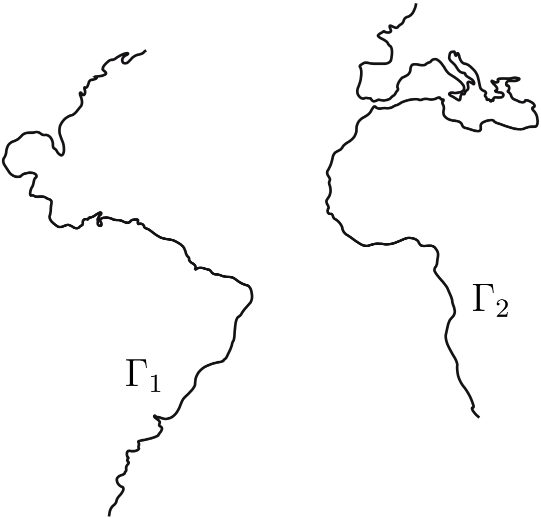
Let us state the main problem.

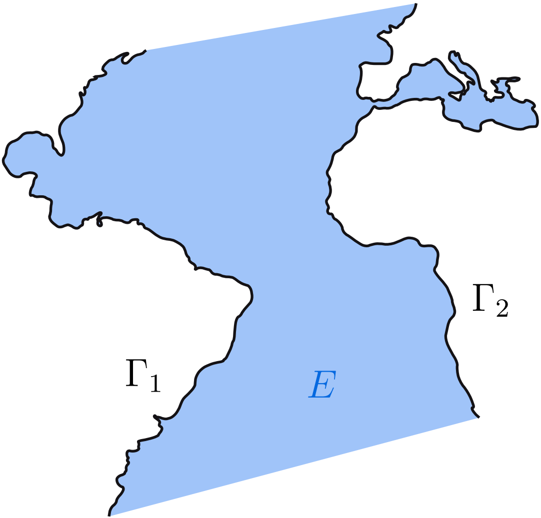


Suppose that we want to reach the shore Γ_1...



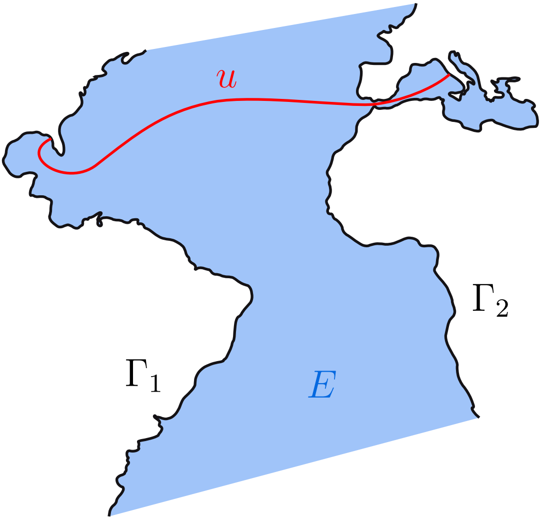
... leaving from our shore Γ_2...

Introduction

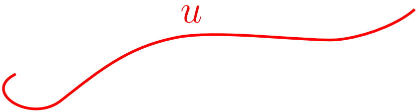


... by crossing the ocean E.

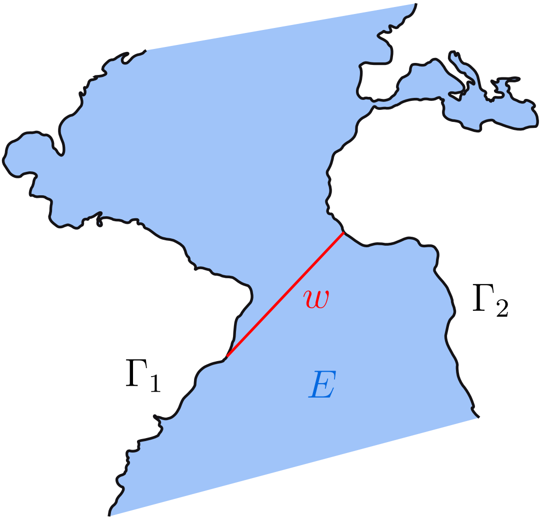
Introduction



With a modern ship we can follow this route.

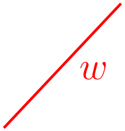


Where u is not a straight line!



Probably ancient sailors

Introduction

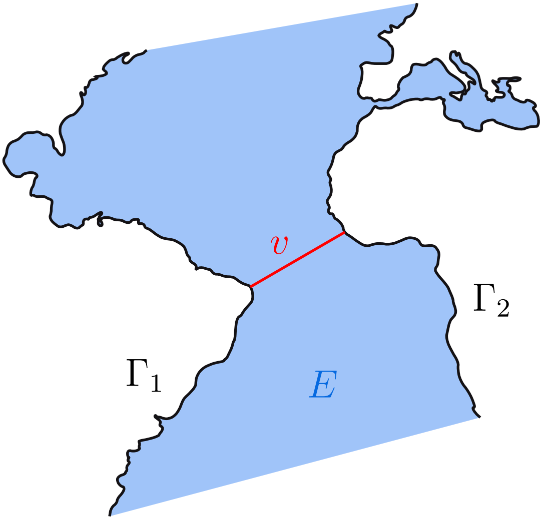


Probably ancient sailors would have followed a straight route w.

Introduction

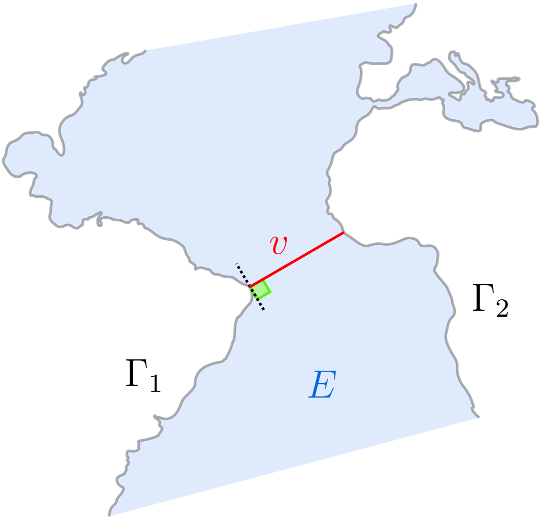
But looking carefully at the map...

Introduction



... it is easy to convince oneself that there exists only one straight route v...

Introduction



... which also ends at a right angle on Γ_1...

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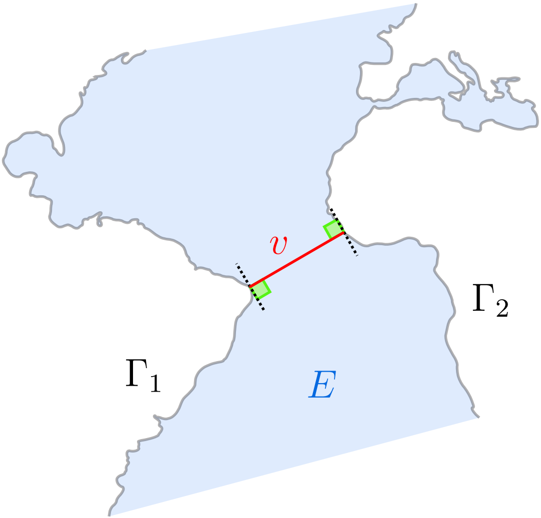
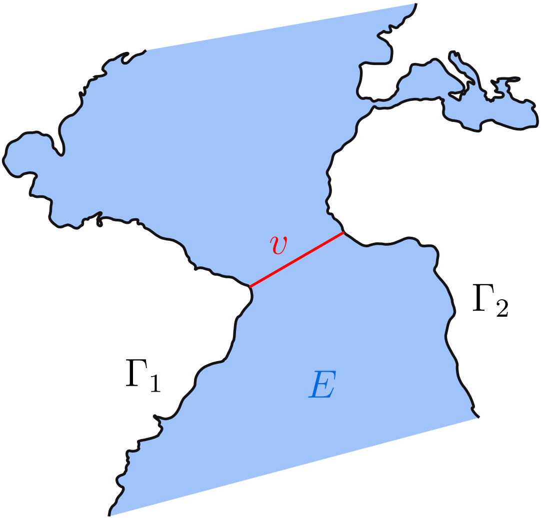


... and also begins at a right angle from Γ_2.



In fact, such a route, is the shortest way!

The mystery of the missing boundary conditions

We have heuristically showed that the

PROBLEM OF COLUMBUS

Given the curves Γ_1 and Γ_2 in $\mathbb{R}^2$, find, among the (non self–intersecting) (smooth) curves which start from a point of Γ_1 and ends to a point of Γ_2 (without crossing $\Gamma_1 \cup \Gamma_2$ in any other point), those whose length is (locally) minimal.

admits a unique solution v, even though the Euler–Lagrange equations associated with the length functional are 2–nd order.

By manipulating the first variational formula, it has been discovered that **the extremal v in fact fulfills some hidden boundary conditions**, which where called transversality conditions.

We will show, in a natural geometric language, that such conditions in fact arise for a large class $\mathcal{P}$ of variational problems. However, we use the problem of Columbus as a **toy model**.

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The largest class of problems encompassed by our theory.

The problem of Columbus is just a naive example of a

FREE BOUNDARY VARIATIONAL PROBLEM $\mathcal{P}$

which can be formalized in the following way: given

- a manifold E with non–empty boundary ∂E,
- an integer $n < \dim E$,
- the set $\mathrm{Adm}(\mathcal{P}) = \{L\}$ such that
  - L is an n–dimensional compact connected submanifolds of E,
  - L is nowhere tangent to ∂E,
  - and ∂L is non–empty and coincides with $L \cap \partial E$,
- and, finally, an horizontal n–form $\omega \in \overline{\Lambda}^n(J^\infty(E, n))$,

we want to find the (local) extrema for the action

$$\mathrm{Adm}(\mathcal{P}) \ni L \mapsto \int_L j_\infty(L)^*(\omega) \in \mathbb{R} \qquad (2)$$

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- a manifold E with non–empty boundary ∂E,
- an integer $n < \dim E$,
- the set $\mathrm{Adm}(\mathcal{P}) = \{L\}$ such that
  - L is an n–dimensional compact connected submanifolds of E,
  - L is nowhere tangent to ∂E,
  - and ∂L is non–empty and coincides with $L \cap \partial E$,
- and, finally, an horizontal n–form $\omega \in \overline{\Lambda}^n(J^\infty(E, n))$,

we want to find the (local) extrema for the action

$$\mathrm{Adm}(\mathcal{P}) \ni L \mapsto \int_L j_\infty(L)^*(\omega) \in \mathbb{R} \qquad (2)$$

determined by ω on the elements of $\mathrm{Adm}(\mathcal{P})$.

The largest class of problems encompassed by our theory.

The problem of Columbus is just a naive example of a

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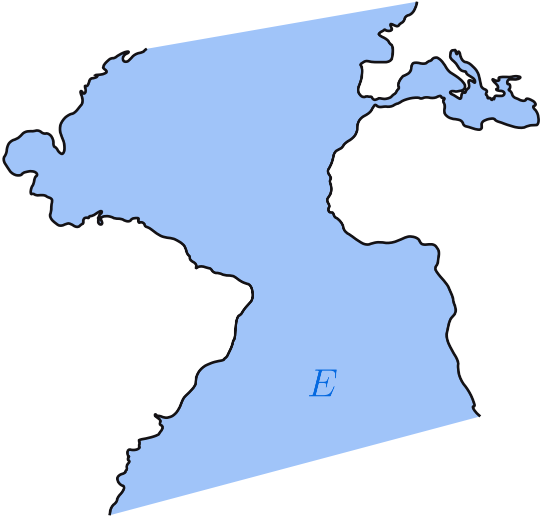
Rectification of the Columbus problem



A nice idea to attack the problem of Columbus would be to transform the ocean E...

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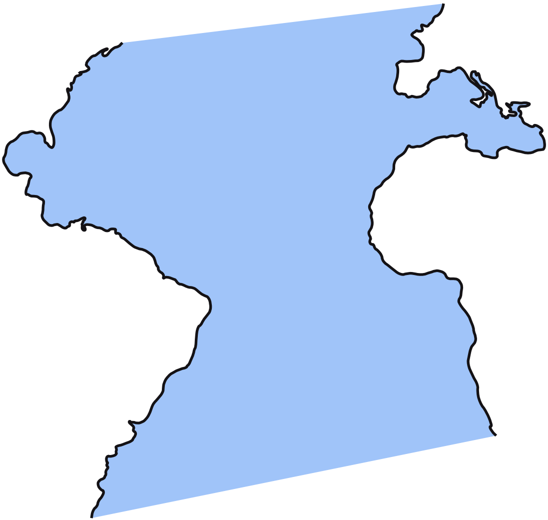

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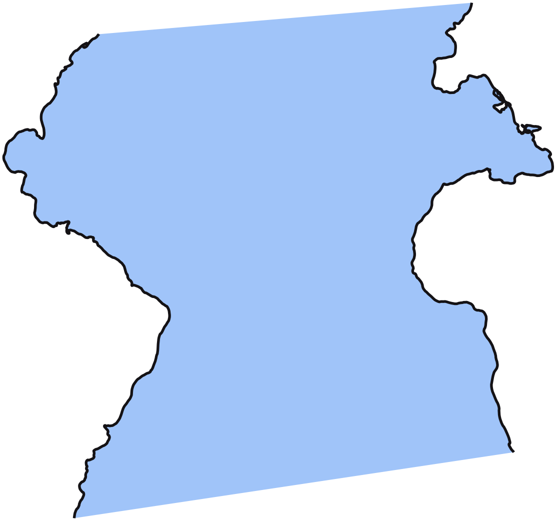

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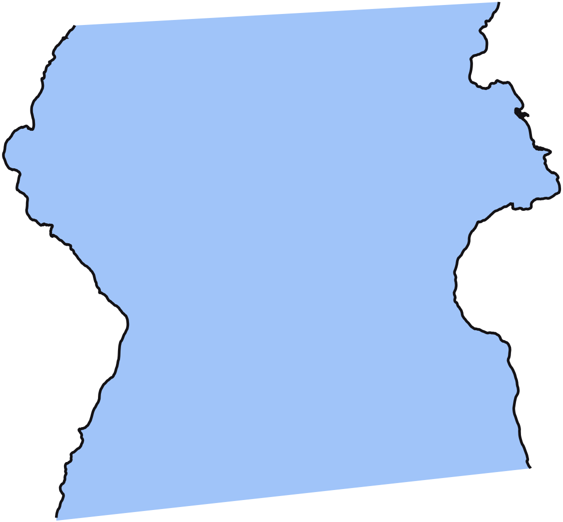

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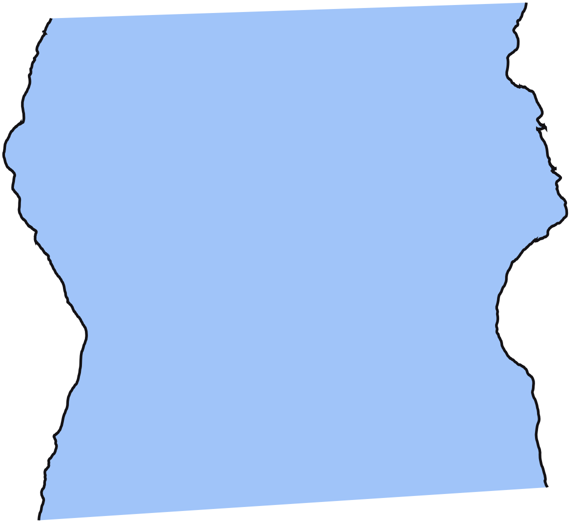

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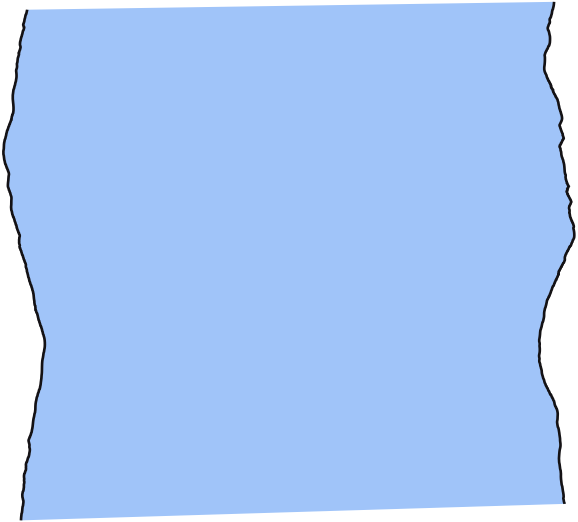

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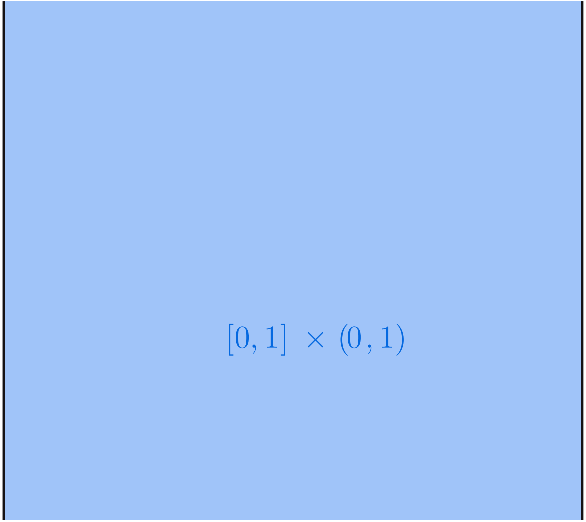

... into the total space of the trivial bundle $\pi : [0,1] \times (0,1) \to [0,1]$.

The rectified problem of Columbus

Since our theory is a natural construction, **the rectified problem of Columbus is equivalent to the original one**, and we can use it as a toy model. Its data are the following:

- $E = [0,1] \times (0,1)$, and $\partial E = \{0,1\} \times (0,1)$,
- $n = 1$,
- $\mathrm{Adm}(\mathcal{P}) = \Gamma(\pi)$
- the horizontal 1–form $\omega \in \overline{\Lambda}^1(J^\infty(\pi))$ is given by $\omega = f(x,y,y')dx$, where $f = \sqrt{1+(y')^2}$,

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Problems $\mathcal{P}$ in which only sections of a fiber bundle π are involved belongs to the so–called **fibered case**.

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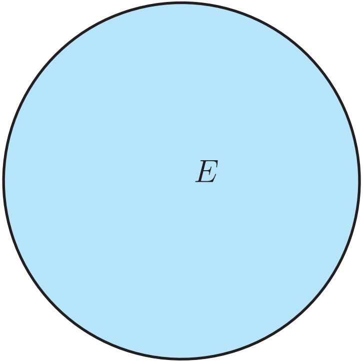
Example of a more general setting

Consider the following problem $\mathcal{P}$:

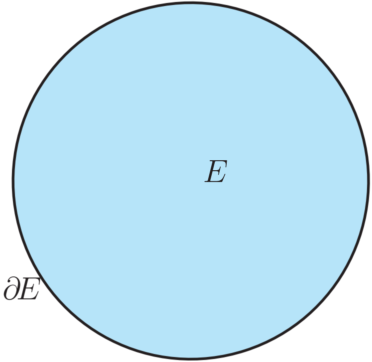
- E is the 2–dimensional closed disk D^2,



Example of a more general setting

Consider the following problem $\mathcal{P}$:

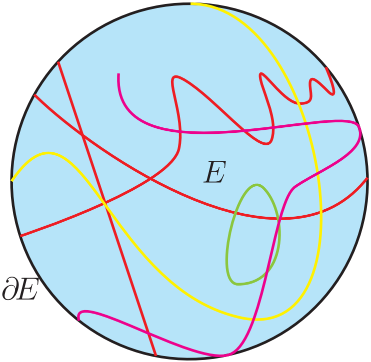
- ∂E is the circle S^1,



Example of a more general setting

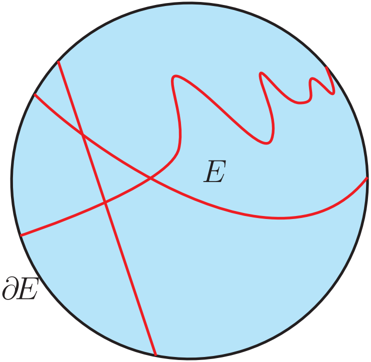
Consider the following problem $\mathcal{P}$:

- and $n = 1$.



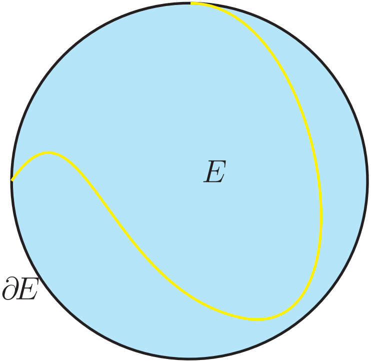
Example of a more general setting

In such a case, the red submanifolds belong to $\mathrm{Adm}(\mathcal{P})$.



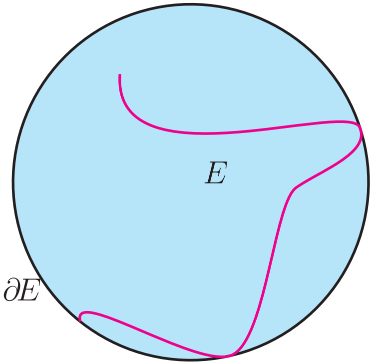
Example of a more general setting

But not the yellow one, for it is tangent to ∂E in some point.



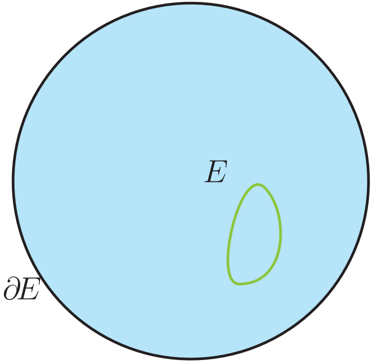
Example of a more general setting

Nor the purple one, since its boundary is not the set of its common points with ∂E.



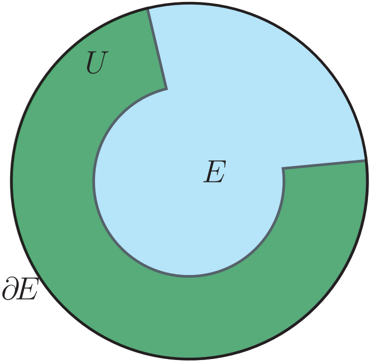
Example of a more general setting

Nor the green one, since it has got empty boundary!



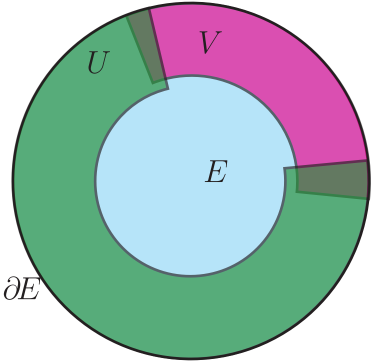
Example of a more general setting

However, **our theory can be localized**. Consider, e.g., the subset U of E.



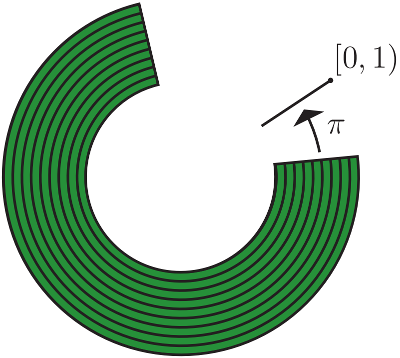
Example of a more general setting

And also the subset V.



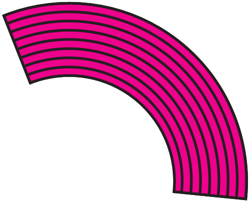
Example of a more general setting

Observe that U has got a fiber bundle structure over $[0, 1)$.



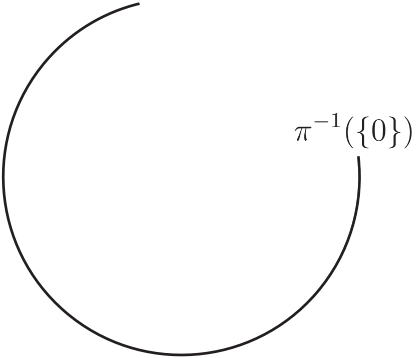
Example of a more general setting

As well as V.



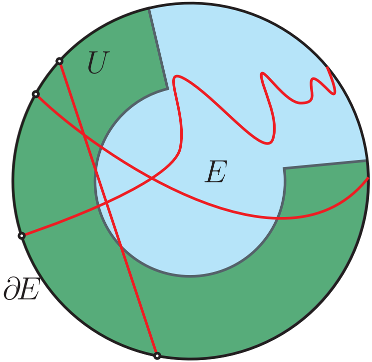
Example of a more general setting

Notice also that $\partial[0,1)$ is just $\{0\}$, so that $\pi^{-1}(\partial[0,1))$ coincides with $U \cap \partial E$.



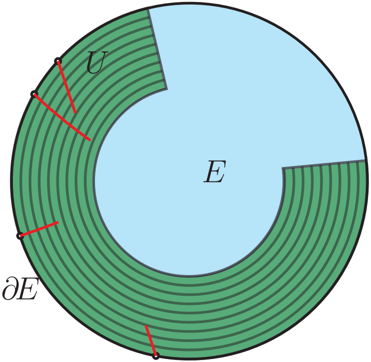
Example of a more general setting

So, nearby a point of ∂E, an admissible submanifold for the problem $\mathcal{P}$...



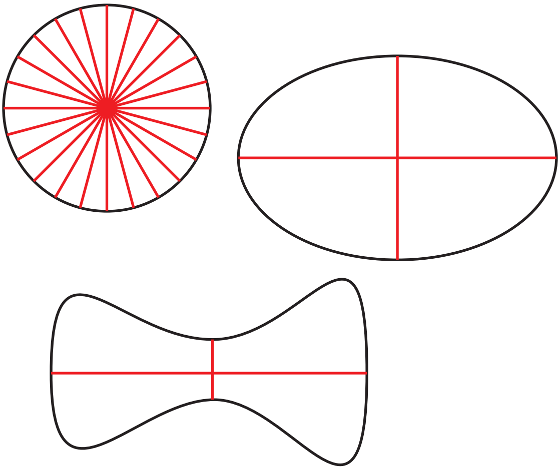
Example of a more general setting

... coincide with the graph of a (local) section of π in a neighborhood of 0.

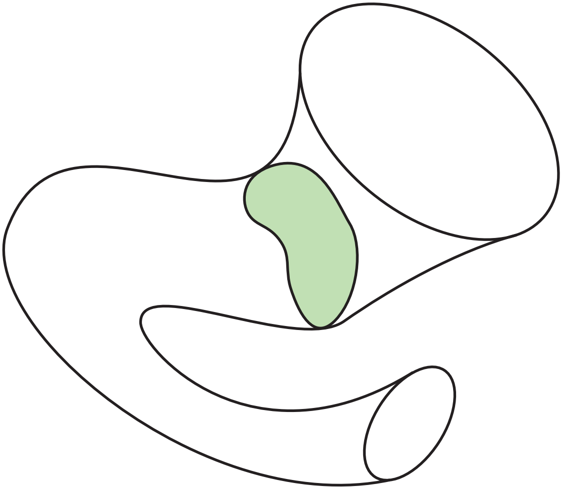


Example of a more general setting

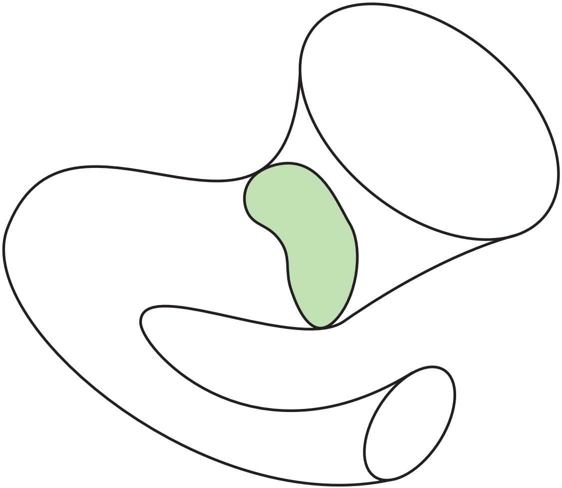
In such a setting, we can, for instance, write down the transversality conditions for the problem of **stationary chords**.



At this point, it should be clear that the rectified problem of Columbus is not such a restrictive toy model, **since by localization and naturality, any problem** $\mathcal{P}$ **can be reduced to the fibered case**.



Now we will show that the theory which describe the problems $\mathcal{P}$ is just an aspect of the Secondary Calculus over a certain diffiety, wich we will call $(B, \mathcal{C})$, naturally associated to the problem $\mathcal{P}$, much as the problem $dF = \omega$ is an aspect of the differential cohomology of $\mathbb{R}^n$.



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We stress that natural here means that if the data of the problem $\mathcal{P}$ undergo a transformation, then the objects and morphisms which constitute the theory which describe $\mathcal{P}$ also undergo a transformation, and the theory which describe the transformed problem is obtained.

What is Secondary Calculus?

The $\mathcal{C}$–spectral sequence is a cohomological theory naturally associated with any space of infinite jets, thanks to the presence of the distribution $\mathcal{C}$, which allows to write down many concepts of the variational calculus by using the same logic of the standard differential calculus.
The resulting language is called Secondary Calculus.

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Our equipment

The secondary objects associated with a free boundary variational problem are:

The main diffiety

$(B, \mathcal{C})$, where $B \stackrel{\text{def}}{=} J^\infty(\pi)$, $\pi : E \longrightarrow M$, and $\mathcal{C}$ is the Cartan distribution.

The sub–diffiety

$(\partial B, \mathcal{C}_{\partial B})$, where $\partial B \stackrel{\text{def}}{=} \pi_\infty^{-1}(\partial M)$ is a sort of "infinite prolongation" of ∂M, and $\mathcal{C}_{\partial B}$ is the restriction of $\mathcal{C}$ to ∂B.

These are the geometrical objects we need to achieve our purpose. But, together with such objects, also comes some interrelationship, which influences the cohomological theory associated with them. This last fact will become clear in a moment.

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The term E_0 of the relative $\mathcal{C}$–spectral sequence

Much as to any submanifold it is possible to associate a short exact sequence of differential forms, to the sub–diffiety ∂B it is possible to associate a

short exact sequence of E_0 terms of $\mathcal{C}$–spectral sequences:

$$0 \rightarrow E_0^p(B, \partial B) \xrightarrow{i} E_0^p \xrightarrow{\alpha} E_0^p(\partial B) \rightarrow 0. \quad (4)$$

The term $E_0^p(B, \partial B)$, which is defined as

$$E_0^p(B, \partial B) = \frac{\mathcal{C}^p \cap \Lambda(B, \partial B) + \mathcal{C}^{p+1}}{\mathcal{C}^{p+1}}, \quad (5)$$

can be understood as the sub–complex of E_0^p whose elements vanish when they are restricted to ∂B, in fully accordance with the definition of a relative (with respect to the boundary) form on a standard manifold. Inspired by this analogy, we call $E_0^p(B, \partial B)$ the E_0 term of the relative (with respect to ∂B) $\mathcal{C}$–spectral sequence associated with B.

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$$E_0^p(B, \partial B) = \frac{\mathcal{C}^p \cap \Lambda(B, \partial B) + \mathcal{C}^{p+1}}{\mathcal{C}^{p+1}}, \quad (5)$$

can be understood as the sub–complex of E_0^p whose elements vanish when they are restricted to ∂B, in fully accordance with the definition of a relative (with respect to the boundary) form on a standard manifold. Inspired by this analogy, we call $E_0^p(B, \partial B)$ the E_0 term of the relative (with respect to ∂B) $\mathcal{C}$–spectral sequence associated with B.

The term E_0 of the relative $\mathcal{C}$–spectral sequence

Much as to any submanifold it is possible to associate a short exact sequence of differential forms, to the sub–diffiety ∂B it is possible to associate a

short exact sequence of E_0 terms of $\mathcal{C}$–spectral sequences:

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But the analogy with standard manifolds goes even further. Analogously to the cohomology long exact sequence, we have the

long exact sequence of E_1 terms:

$$
\begin{array}{ccc}
E_1^p(B, \partial B) & \xrightarrow{H(i)} & E_1^p \\
 & {\scriptstyle \partial} \nwarrow \quad \swarrow {\scriptstyle H(\alpha)} & \\
 & E_1^p(\partial B). &
\end{array}
\tag{6}
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The new object here is $E_1^p(B, \partial B)$, the E_1 term of the relative $\mathcal{C}$–spectral sequence, which is the $\mathcal{C}$–spectral sequence appearing in the title of this talk!

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The equipment is complete

All the objects needed to build up the theory, are now ready. Observe that, together with few new objects (the 0–th and the first terms of the relative $\mathcal{C}$–spectral sequence), we have also introduced (in a more or less explicit way) new differentials and morphisms.
In a moment, we shall see how such objects and differentials and morphisms can encode all the known information about the problem $\mathcal{P}$, and also reveal some new aspects.

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The right definition of the Lagrangian

In some sense, the whole theory of the free boundary variational problems depends on a good definition of the Lagrangian. All the rest, is an (almost) algorithmic consequence of this definition.

In the statement of $\mathcal{P}$, only the Lagrangian density ω was involved. Observe that, for any $u \in \Gamma(\pi)$, the map $j_\infty(u)$ sends ∂M into ∂B. So, if ω is the differential of some form vanishing on ∂B, then $j_\infty(u)^*(\omega)$ will be the differential of some form vanishing on ∂M. By Stokes formula, the action determined by ω, evaluated on M, is zero.

In other words, the action of ω is given only by its cohomology class modulo ∂B. Then we can say that the

Lagrangian associated with $\mathcal{P}$

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The relative (or “graded”) Euler–Lagrange equations

In the language of Secondary Calculus, all the information about $\mathcal{P}$ is kept by the element $[\omega]$ of $E_1^{0,n}(B, \partial B)$.

Moreover, the logic of Secondary Calculus dictates the procedure to get the equation for the extrema of $\mathcal{P}$: just apply the differential $d_{1,\mathrm{rel}}^{0,n}$ to $[\omega]$, much as we apply the standard differential d to a function f on a smooth manifold M to get the equation for the extrema of f.

This way we discovered the

(left–hand side of the) relative (or “graded”) Euler–Lagrange equations associated with $\mathcal{P}$:

$$d_{1,\mathrm{rel}}^{0,n}([\omega]) \in E_1^{1,n}(B, \partial B). \tag{7}$$

Good news: if we “look inside” $d_{1,\mathrm{rel}}^{0,n}([\omega])$, we can see both the (standard) Euler–Lagrange equations, and the (generalized) transversality conditions.

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Good news: if we "look inside" $d_{1,\mathrm{rel}}^{0,n}([\omega])$, we can see both the (standard) Euler–Lagrange equations, and the (generalized) transversality conditions.

Finally, Secondary Calculus assures us that the relative (or “graded”) Euler–Lagrange equation

$$d_{1,\mathrm{rel}}^{0,n}([\omega]) = 0 \tag{8}$$

is satisfied by the extrema of $\mathcal{P}$.

But, how can we call (8) an “equation”?

In general, $E_1^{1,n}(B, \partial B)$ is not even a module, so $d_{1,\mathrm{rel}}^{0,n}([\omega])$ cannot be interpreted as a differential operator.

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The one–line Theorem for ∂B

We must now investigate the structure of ∂B, which is revealed by the following

Structure Theorem (G.M.)

$\pi^{-1}(\partial B)$ is isomorphic to the infinite jet space $J^\infty(\xi)$, where ξ is a special ∞–dimensional bundle over ∂M, called the normal jets bundle.

As a consequence, the diffiety ∂B fulfills the one–line Theorem (A.M. Vinogradov).

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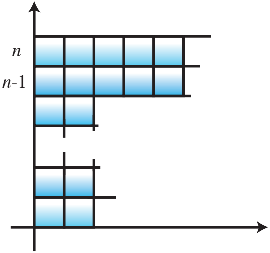
The one–line Theorem for the relative $\mathcal{C}$–spectral sequence

Now, if we just rewrite the long exact sequence

$$\begin{array}{ccc} E_1^p(B,\partial B) & \xrightarrow{H(i)} & E_1^p \\ & {\scriptstyle \partial}\nwarrow \quad \swarrow{\scriptstyle H(\alpha)} & \\ & E_1^p(\partial B). & \end{array}$$

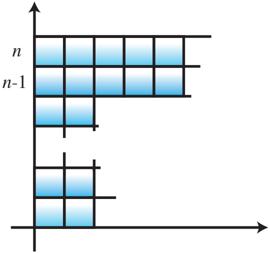
in a more choreographic way...

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



…first the relative E_1 term…

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



…then the main E_1 term…

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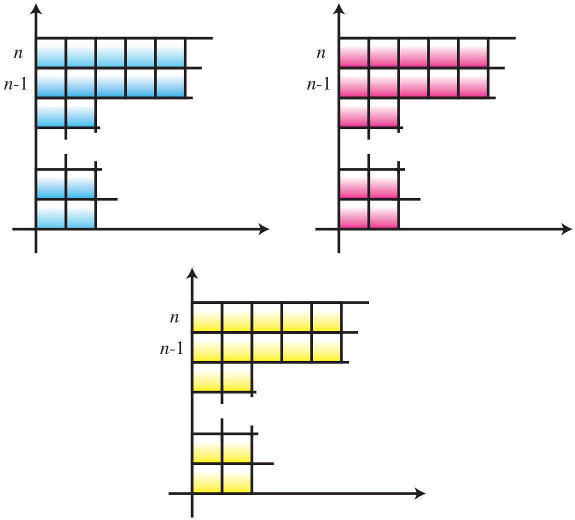
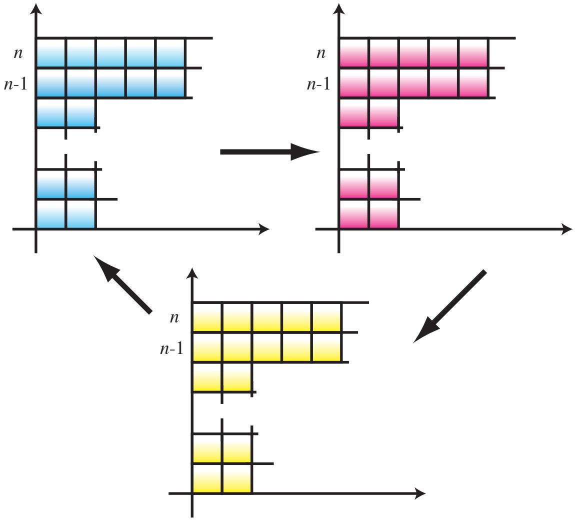


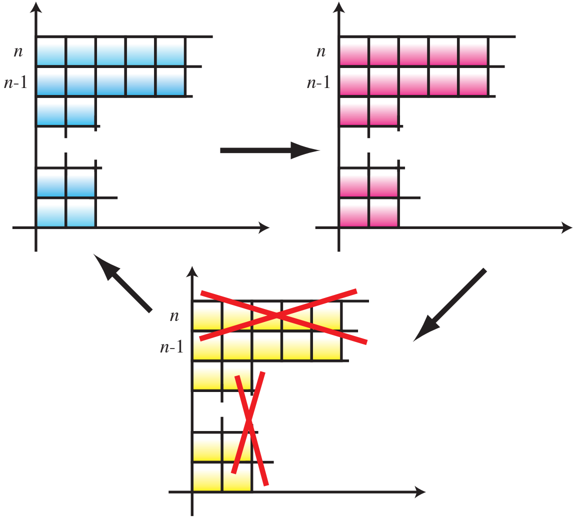
…and finally the E_1 term of ∂B,

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



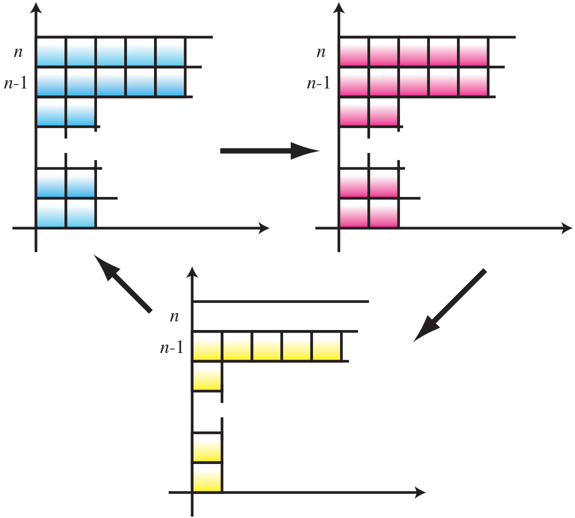
we get a better comprehension of this short exact sequence of E_1 terms.

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



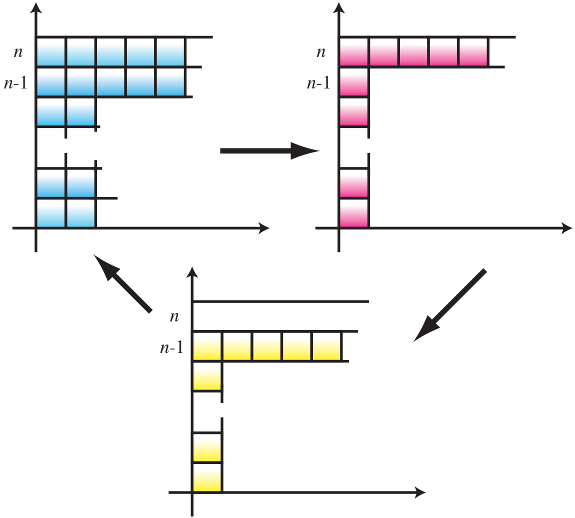
Indeed, thanks to the Structure Theorem...

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



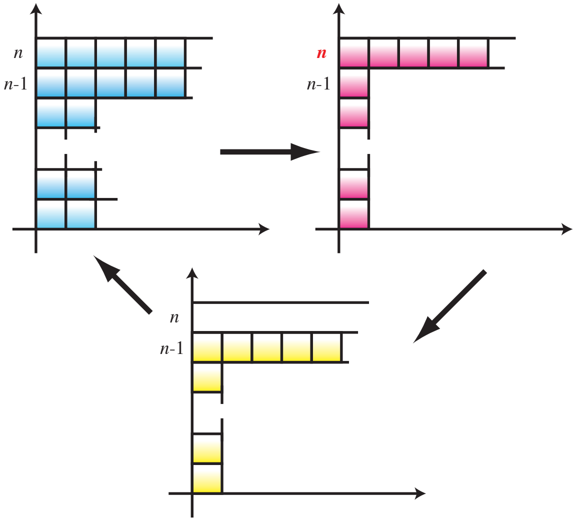
...the one–line Theorem holds for ∂B.

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



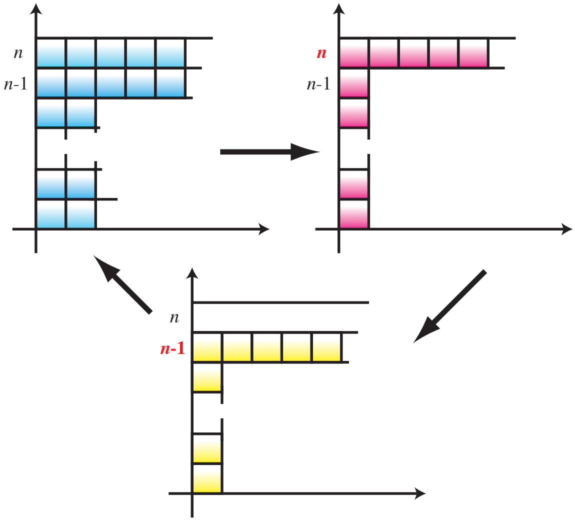
It is well–known that the one–line Theorem holds for B.

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



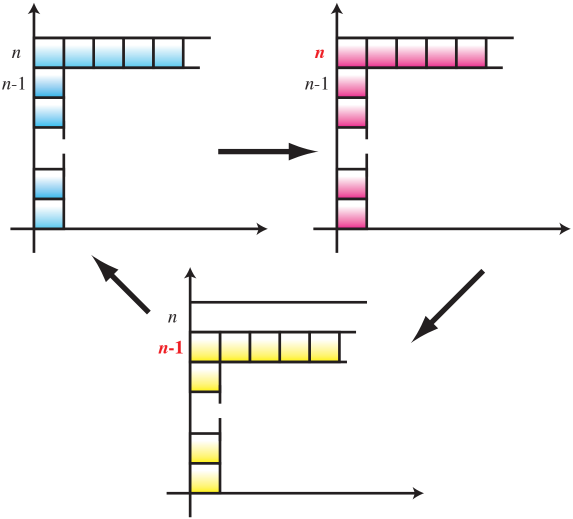
Where in B we have n independent variables...

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



...while in ∂B they are just $n-1$.

The one–line Theorem for the relative $\mathcal{C}$–spectral sequence



However, taking into account the degrees of the maps into play, we see that also the relative spectral sequence is forced to be one–line!

The short exact sequence of E_1 terms

In other words, we have found the

short exact sequence of E_1 terms

$$0 \longrightarrow \widehat{\varkappa}(\partial B) \xrightarrow{\partial} E_1^{1,n}(B, \partial B) \xrightarrow{H(i)} \widehat{\varkappa} \longrightarrow 0. \quad (9)$$

By using the so–called Green $\mathcal{C}$–formula, a technical aspect of the cohomological theory underlying the Secondary Calculus, it is easy to prove that the above sequence, in fact, splits.

Such a splitting is crucial for obtaining the vision of $d_{1,\mathrm{rel}}^{0,n}([\omega])$ we were looking for.

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The graded Euler–Lagrange equations

Now we know that the graded Euler–Lagrange equations $d_{1,\mathrm{rel}}^{0,n}([\omega])$ look like an element

$$(\ell'_\omega, \ell^*_\omega(1))$$

of the graded object $\widehat{\varkappa}(\partial B) \oplus \widehat{\varkappa}$.
The second component is the (left–hand side of the) well–known Euler–Lagrange equations associated to ω.

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The second component is the (left–hand side of the) well–known Euler–Lagrange equations associated to ω.

The (generalized) transversality conditions

The first component of $d_{1,\mathrm{rel}}^{0,n}([\omega])$ is a new object, which we baptized "the (left–hand side of the) transversality conditions" associated with $\mathcal{P}$, to be consistent with the already established terminology.

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Thanks to the Structure Theorem, ℓ'_ω is an element of a vector bundle over ∂M. As such, it will admit a coordinate expression, which we show below, though quite complicated:

$$\sum_{j=1}^{m}\sum_{k=0}^{\infty}\sum_{\tau\in\mathbb{N}_0^{n-1}}\sum_{l=k}^{\infty} N(\tau,l,i)D_\tau\left(\left(D_n^{l-k}\left(\frac{\partial f}{\partial u^j_{\tau+(l+1)1_n}}\right)\right)\Big|_{\partial B}\right)D_\emptyset^{(k,j)}$$

where m is the dimension of π.

Here $fdx^1\wedge\cdots\wedge dx^n$ is the local representation of ω, and $x^n=0$ is the equation for ∂M. The D's are (compositions of) the total derivatives.

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The rectified problem of Columbus

If the Lagrangian density on E is given by fdx, where $f = f(x, y, y')$, then the corresponding Lagrangian on π will be given by $\omega = gdx$, for some function $g = g(x, y, y')$.

So we can apply the coordinate formula: it tells that ℓ'_ω is simply $\left.\frac{\partial g}{\partial y'}\right|_{\partial B}$, so that the transversality conditions (accordingly to our theory) look like:

$$\left.\frac{\partial g}{\partial y'}\right|_{\partial B} = 0. \tag{10}$$

If we pull–back the last expression on E, we get the following expression:

$$\left(f - \frac{\partial f}{\partial y'}\right) x^\Gamma + \frac{\partial f}{\partial y'} y^\Gamma = 0, \tag{11}$$

where (x^Γ, y^Γ) is a tangent to ∂E vector, which is the classical formulation of the transversality conditions.

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The rigorous solution of the Columbus problem

In the particular case of Columbus problem, $f = \sqrt{1+(y')^2}$ is the (restriction to E of the) length functional (on $\mathbb{R}^2$), and the last equation tells exactly that the curve u must form a right angle with Γ_1 and Γ_2 (TC).

On the other hand, $\ell^*_{fdx}(1) = 0$ is equivalent to $y'' = 0$ (EL).

So, conditions (TC) and (EL), of such heterogeneous natures (they are differential equations imposed on sections of bundles over different bases and with non–isomorphic fibers!), are in fact the two homogeneus components of the same graded object: $d_{1,\mathrm{rel}}^{0,1}([fdx])$.

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Concluding remarks

Before this moment, the formula

$$\left(f - \frac{\partial f}{\partial y'} \right) x^{\Gamma} + \frac{\partial f}{\partial y'} y^{\Gamma} = 0, \tag{12}$$

which we obtained just by using the coordinates–invariance of a purely cohomological theory, could not be derived without introducing ad–hoc technicalities.

In our case, thanks to the robustness of Secondary Calculus, we have managed to provide a simple description for any problem $\mathcal{P}$ which belongs to much more wide and general class of problems, where we have Lagrangians of any order, and no restrictions on the topology of E.

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Thanks!

See You at the next Conference.