New results on the geometry of translation surfaces

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Outline

1. Translation surfaces in $\mathbb{E}^3$

2. On the geometry of the second fundamental form of translation surfaces in $\mathbb{E}^3$
   - $\{K_{II}, H\}$ - Generalized Weingarten translation surfaces
   - $II$-minimality

3. Translation surfaces in the hyperbolic space $\mathbb{H}^3$

4. Translation surfaces in the Heisenberg group $\mathbb{Nil}_3$

5. Translation surfaces in $\mathbb{S}^3$

6. Final remarks
Darboux surfaces

Cartesian parametrization:

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = A(v) \begin{pmatrix}
    f(u) \\
    g(u) \\
    h(u)
\end{pmatrix} + \begin{pmatrix}
    a(v) \\
    b(v) \\
    c(v)
\end{pmatrix}
\]

where \( A(v) \in O(n) \)
Darboux surfaces

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\]

where \( A(v) \in O(n) \)

A Darboux surface represents a union of "EQUAL" curves (i.e. the image of one curve\(^1\), obtained by isometries of the space.

\(^1\) generatrix


**Darboux surfaces**

1. $A = I_3$: translation surfaces
Darboux surfaces

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2. $A = \text{matrix of rotation (axe and angle are fixed), } a = b = c = 0$: rotation surfaces
Darboux surfaces

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2. $A =$ matrix of rotation (axe and angle are fixed), $a = b = c = 0$: rotation surfaces
3. $A =$ matrix of rotation (axe $\bar{n}$ and angle are fixed), $(a, b, c) = v\bar{n}$: helicoidal surfaces
Darboux surfaces

1. $A = I_3$: translation surfaces

2. $A =$ matrix of rotation (axe and angle are fixed), $a = b = c = 0$: rotation surfaces

3. $A =$ matrix of rotation (axe $\vec{n}$ and angle are fixed), $(a, b, c) = v\vec{n}$: helicoidal surfaces

If the generatrix is
- a straight line: ruled surfaces
- a circle: circled surfaces including e.g. tubes
Translation surfaces in $\mathbb{E}^3$

Tubes

$$r(s, t) = \gamma(t) + \cos s N(t) + \sin s B(t)$$

Figure: tube
Tubes

\[ r(s, t) = \gamma(t) + \cos s N(t) + \sin s B(t) \]

**Figure:** tube

\[ r(s, t) = \gamma(t) + A(t) S^1 \]
Translation surfaces

Translation surface = ”sum” of two curves

Figure: translation surface
Translation surfaces

If the two curves are situated in orthogonal planes

\[(x, y, z) \mapsto (x, y, f(x) + g(y))\]

Examples:

1. Planes
2. Cylinders
3. Hyperbolic and elliptic paraboloids
4. The egg box surface
5. Scherk surface
Translation surfaces

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3. hyperbolic and elliptic paraboloids
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5. Scherk surface
Egg box surfaces

\[(x, y, a(\sin \frac{x}{b} + \sin \frac{y}{b}))\]

Figure: egg box surface
Scherk surfaces

\[(x, y, a \log \frac{\cos \frac{x}{a}}{\cos \frac{y}{a}})\]

Figure: Scherk surface
Scherk surface - art

... much more beautiful

Figure: Scherk surface
Second fundamental form

ON THE GEOMETRY OF THE SECOND FUNDAMENTAL FORM OF TRANSLATION SURFACES IN $\mathbb{E}^3$


$M$ surface in $\mathbb{E}^3$

$I = \text{the first fundamental form} - \text{intrinsic object}$

$II = \text{the second fundamental form} - \text{extrinsic tool to characterize the twist of } M \text{ in the ambient}$
ON THE GEOMETRY OF THE SECOND FUNDAMENTAL FORM OF TRANSLATION SURFACES IN $\mathbb{E}^3$


$M$ surface in $\mathbb{E}^3$

$I =$ the first fundamental form – intrinsic object

$II =$ the second fundamental form – extrinsic tool to characterize the twist of $M$ in the ambient

$II$ is a metric if and only if it is non-degenerate

curvature properties associated to $II$:

Lemma (Dillen, Sodsiri - 2005)

The second fundamental form $II$ of $M$ is non-degenerate if and only if $M$ is non-developable.
Second fundamental form

Lemma (Dillen, Sodsiri - 2005)

The second fundamental form $\mathbb{II}$ of $M$ is non-degenerate if and only if $M$ is non-developable.

second Gaussian curvature $K_{\mathbb{II}} \implies \mathbb{II}$-flat
second mean curvature $H_{\mathbb{II}} \implies \mathbb{II}$-minimal
Second fundamental form

Lemma (Dillen, Sodsiri - 2005)

The second fundamental form \( II \) of \( M \) is non-degenerate if and only if \( M \) is non-developable.

second Gaussian curvature \( K_{II} \)  \( \implies \) \( II \)-flat
second mean curvature \( H_{II} \)  \( \implies \) \( II \)-minimal

Remark (Verpoort - 2008)

Critical points of the area functional of the second fundamental form are those surfaces for which the mean curvature of the second fundamental form \( H_{II} \) vanishes.
Old results

- **Koutroufiotis - 1974**: a closed ovaloid with \( K_{II} = cK \), \( c \in \mathbb{R} \) or if \( K_{II} = \sqrt{K} \) is a sphere
Old results

- **Koutroufiotis - 1974**: a closed ovaloid with $K_{ll} = cK$, $c \in \mathbb{R}$ or if $K_{ll} = \sqrt{K}$ is a sphere

- **Koufogiorgos & Hasanis - 1977**: the sphere is the only closed ovaloid satisfying $K_{ll} = H$
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- **Koufogiorgos & Hasanis - 1977**: the sphere is the only closed ovaloid satisfying $K_{II} = H$.
- **Baikoussis & Koufogiorgos - 1997**: helicoidal surfaces with $K_{II} = H^{(locally)} \Leftrightarrow$ constant ratio of the principal curvatures.
Old results

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- **Baikoussis & Koufogiorgos - 1997**: helicoidal surfaces with $K_{II} = H$ (locally) constant ratio of the principal curvatures
- **Blair & Koufogiorgos - 1992**: minimal surfaces have vanishing second Gaussian curvature but not conversely
Old and recent results

- **Koutroufiotis - 1974**: a closed ovaloid with $K_{\|} = cK$, $c \in \mathbb{R}$ or if $K_{\|} = \sqrt{K}$ is a sphere.

- **Koufogiorgos & Hasanis - 1977**: the sphere is the only closed ovaloid satisfying $K_{\|} = H$.

- **Baikoussis & Koufogiorgos - 1997**: helicoidal surfaces with $K_{\|} = H$ (locally) constant ratio of the principal curvatures.

- **Blair & Koufogiorgos - 1992**: minimal surfaces have vanishing second Gaussian curvature but not conversely.

**Kim & Yoon - 2004, Sodsiri - 2005, Yoon - 2006** extends the study for 3-dimensional Lorentz-Minkowski spaces and for different relations between $H, K, H_{\|}$ and $K_{\|}$.
Theorem (Goemans, Van de Woestyne - 2007)

If a translation surface in $\mathbb{E}^3_1$ parametrized by $\bar{x}(s, t) = (s, t, f(s) + g(t))$ has $K_{II} = 0$, then

$$f(s) = \int F^{-1}(s + d)ds \quad \text{and} \quad g(t) = \int G^{-1}(t + m)dt$$

with $F$ and $G$ real functions determined by

$$F(x) = \int \frac{x^2}{ax^4 + bx^2 + c} dx \quad \text{and} \quad G(x) = \int \frac{x^2}{-ax^4 + (2a+b)x^2 - a - b - c} dx,$$

and $a, b, c, d \text{ and } m$ real numbers.
**II-flat PT surfaces in \( \mathbb{E}^3 \)**

**polynomial translation surfaces** (in short, *PT surfaces*): translation surfaces for which \( f \) and \( g \) are polynomials

**Theorem (M., Nistor - 2009)**

There are no II-flat polynomial translation surfaces in \( \mathbb{E}^3 \).

**Proof.**

\[
K_{ll} = \frac{1}{(|eg| - f^2)^2} \begin{vmatrix}
-\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\
\frac{1}{2}e_v & e & f \\
\frac{1}{2}g_u & f & g
\end{vmatrix} - \begin{vmatrix}
0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\
\frac{1}{2}e_v & e & f \\
\frac{1}{2}g_u & f & g
\end{vmatrix}
\]
(cont.)

\[ K_{II} = \frac{\text{num}}{4\alpha' \beta' \Delta^{3/2}} \]

where

\[ \text{num} = -2\alpha(u)^2\alpha'(u)^2\beta'(v) - 2\alpha'(u)\beta(v)^2\beta'(v)^2 + 
2\alpha(u)^2\alpha'(u)\beta'(v)^2 + 2\alpha'(u)^2\beta(v)^2\beta'(v) + 
2\alpha'(u)\beta'(v)^2 + 2\alpha'(u)^2\beta'(v) + 
\alpha'(u)\beta(v)\beta''(v) + \alpha(u)\alpha''(u)\beta'(v) + 
\alpha(u)^2\alpha'(u)\beta(v)\beta''(v) + \alpha(u)\alpha''(u)\beta(v)^2\beta'(v) + 
\alpha'(u)\beta(v)^3\beta''(v) + \alpha(u)^3\alpha''(u)\beta'(v). \]
example given by Blair & Koufogiorgos - 1992: Il-flat non-minimal translation surfaces, involving *power functions*, i.e.

\[ \alpha = au^p \text{ and } \beta = bv^q \text{ with } a, b \in \mathbb{R}, \ a, b \neq 0 \text{ and } p, q \in \mathbb{Q}. \]

**Proposition (M., Nistor - 2009)**

The only Il-flat translation surfaces with \( f \) and \( g \) power functions can be parametrized by

\[ r(u, v) = \left( u, v, c(u^\frac{4}{3} - v^\frac{4}{3}) \right), \ c \in \mathbb{R}^*. \]
Translation surfaces in $\mathbb{E}^3$ \{ $K_{II}$, $H$ \} - Generalized Weingarten translation surfaces

$K_{II} = H$

\{ $A$, $B$ \} - generalized Weingarten surfaces: Dillen, Sodsiri - 2005
\( K_{II} = H \)

\{A, B\} - generalized Weingarten surfaces : Dillen, Sodsiri - 2005

**Theorem (M., Nistor - 2009)**

The only translation surfaces with non-degenerate second fundamental form having the property \( K_{II} = H \) are given, up to a rigid motion of \( \mathbb{R}^3 \), by

\[
  r(u, v) = \left( u, \ v, \ \frac{2}{c} \ \log \left| \frac{\cos \frac{cu}{2}}{\cos \frac{cv}{2}} \right| \right), \ c \in \mathbb{R}^*.
\]

More, we notice the parametrization of a Scherk type surface, so we have

\[
  K_{II} = H = 0.
\]
Translation surfaces in $\mathbb{E}^3$

\[ K_{\|} = \lambda H, \; \lambda \neq 1, 2 \]

Theorem (M., Nistor - 2009)

The only $\{K_{\|}, H\}$–generalized Weingarten translation surfaces with non-degenerate second fundamental form satisfying $K_{\|} = \lambda H$ with $\lambda \in \mathbb{R} \setminus \{1, 2\}$, are given, up to a rigid motion of $\mathbb{R}^3$, by the parametrization

\[
r(u, v) = \left( u, v, \frac{1}{p} \log \left| \frac{\cos(pv + r)}{\cos(pu + q)} \right| \right), \quad \text{where } p \neq 0 \text{ and } r, q \in \mathbb{R}
\]

which represents a Scherk type surface. Moreover $K_{\|} = H = 0$. 
\[ K_{II} = 2H \]

**Theorem (M., Nistor - 2009)**

The only translation surfaces with non-degenerate second fundamental form having the property \( K_{II} = 2H \) are given, up to a rigid motion of \( \mathbb{R}^3 \), by the following parametrizations

1) Case 1.

\[
\begin{align*}
  r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \left( \sinh(pu)^{\frac{1}{p^2}} \cos(qv)^{\frac{1}{q^2}} \right) \right) \\
  r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \left( \cosh(pu)^{\frac{1}{p^2}} \cos(qv)^{\frac{1}{q^2}} \right) \right)
\end{align*}
\]

Case 2.

\[
\begin{align*}
  r(u, v) &= \left( u, v, \frac{\nu}{2} \log \frac{\cos(pu)^{\frac{1}{p^2}}}{\cos(qv)^{\frac{1}{q^2}}} \right)
\end{align*}
\]
**$K_{II} = 2H$**

**i)** Case 3.

\[
\begin{align*}
r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \frac{\sinh(\rho u)}{\sinh(\rho v)} \right) \\
r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \frac{\cosh(\rho u)}{\cosh(\rho v)} \right)
\end{align*}
\]

\[
\begin{align*}
r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \frac{\cosh(\rho u)}{\sinh(\rho v)} \right) \\
r(u, v) &= \left( u, v, -\frac{\nu}{2} \log \frac{\sinh(\rho u)}{\cosh(\rho v)} \right)
\end{align*}
\]

**ii)**

\[r(u, v) = (u, v, a(u - u_0)^2 - a(v - v_0)^2), \quad a, u_0, v_0 \in \mathbb{R}\]

hyperbolic paraboloid.

**iii)** combinations of the previous functions in (i) and a second order polynomial (as in (ii), for a certain $a$)
Translation surfaces in $\mathbb{E}^3$ \{K, H\} - Generalized Weingarten translation surfaces

Figures

\[ r(u, v) = (u, v, \log(\sinh u \cos v)) \]
\[ r(u, v) = (u, v, \log(\cosh u \cos v)) \]
Translation surfaces in $\mathbb{E}^3$ \{ $K_\parallel$, $H$ \} - Generalized Weingarten translation surfaces

Figures

\[ r(u, v) = \left( u, v, \log \frac{\cosh u}{\cosh v} \right) \]

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Marian Ioan MUNTEANU (UAIC)  On the geometry of translation surfaces  Varna, June 2009 23 / 39
$\|\text{-minimal surfaces}$

Haesen, Verpoort, Verstraelen - 2008

\[
H_{\|} = -H - \frac{1}{4} \Delta^\| \log(K)
\]

where $\Delta^\|$ is the Laplacian for functions computed with respect to the second fundamental form as metric. $H_{\|}$ can be equivalently expressed as

\[
H_{\|} = -H - \frac{1}{2 \sqrt{\det \|}} \sum_{i,j} \frac{\partial}{\partial u^i} \left( \sqrt{\det \|} h^{ij} \frac{\partial}{\partial u^j} (\log \sqrt{K}) \right).
\]
II-minimal translation surfaces

\[(u, v) \mapsto (u, v, f(u) + g(v)); \alpha = f', \beta = g'\]

\[H_{II} = 0\] is equivalent to

\[
\frac{(1 + \alpha^2)\beta' + (1 + \beta^2)\alpha' - 4}{(1 + \alpha^2 + \beta^2)^2} + \frac{\alpha'''\alpha' - 2\alpha''^2}{2\alpha'^4} + \frac{\beta'''\beta' - 2\beta''^2}{2\beta'^4} = 0
\]
II-minimal translation surfaces

\[(u, v) \mapsto (u, v, f(u) + g(v)); \alpha = f', \beta = g'\]

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\[
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\]

After STRAIGHTFORWARD COMPUTATIONS it follows
\[\alpha' = 0, \beta' = 0\] which cannot occur since II is no longer invertible
II-minimal translation surfaces

$$(u, v) \mapsto (u, v, f(u) + g(v)); \alpha = f', \beta = g'$$

$H_{II} = 0$ is equivalent to

$$\frac{(1 + \alpha^2)\beta' + (1 + \beta^2)\alpha' - 4}{(1 + \alpha^2 + \beta^2)^2} + \frac{\alpha'''\alpha' - 2\alpha''^2}{2\alpha'^4} + \frac{\beta'''\beta' - 2\beta''^2}{2\beta'^4} = 0$$

After STRAIGHTFORWARD COMPUTATIONS it follows

$\alpha' = 0$, $\beta' = 0$ which cannot occur since $II$ is no longer invertible

Theorem (M., Nistor - 2009)

There are NO II-minimal translation surfaces in Euclidean 3-space.
General things


\( \mathbb{H}^3 \) hyperbolic space: upper half-space \( \mathbb{R}^3_+ \)

\[ ds^2 = \frac{1}{z^2} (dx^2 + dy^2 + dz^2) \]
General things


$\mathbb{H}^3$ hyperbolic space: upper half-space $\mathbb{R}^3_+$

$ds^2 = \frac{1}{z^2} \left( dx^2 + dy^2 + dz^2 \right)$

The absence of an affine structure does not permit to give an intrinsic concept of translation surface as in $\mathbb{E}^3$ $\Rightarrow$ sum of planar curves
General things


\( \mathbb{H}^3 \) hyperbolic space : upper half-space \( \mathbb{R}^3_+ \)

\[ ds^2 = \frac{1}{z^2} (dx^2 + dy^2 + dz^2) \]

the absence of an affine structure does not permit to give an intrinsic concept of translation surface as in \( \mathbb{E}^3 \) \( \rightarrow \) sum of planar curves

\( x, y \) are interchangeable, but not with \( z \)

**type 1**: \( r(x, y) = \{x, y, f(x) + g(y)\} \)

**type 2**: \( r(x, z) = \{x, f(x) + g(z), z\} \)
General things


\( \mathbb{H}^3 \) hyperbolic space : upper half-space \( \mathbb{R}^3_+ \)

\[ ds^2 = \frac{1}{z^2} (dx^2 + dy^2 + dz^2) \]

The absence of an affine structure does not permit to give an intrinsic concept of translation surface as in \( \mathbb{E}^3 \) \( \rightarrow \) sum of planar curves

\( x, y \) are interchangeable, but not with \( z \)

**type 1** : \( r(x, y) = \{ x, y, f(x) + g(y) \} \)

**type 2** : \( r(x, z) = \{ x, f(x) + g(z), z \} \)

Notice that there are NO isometries of \( \mathbb{H}^3 \) that carry surfaces of type 1 into surfaces of type 2 or vice-versa.
Minimal translation surface

Recall: in $\mathbb{E}^3 \rightarrow$ planes and Scherk surface

Known fact: Examples of minimal surfaces in $\mathbb{H}^3$: totally geodesic planes, minimal graphs (corresponding to Dirichlet problem)
Minimal translation surface

Recall: in $\mathbb{E}^3$ planes and Scherk surface

Known fact: Examples of minimal surfaces in $\mathbb{H}^3$: totally geodesic planes, minimal graphs (corresponding to Dirichlet problem)

Theorem (López - 2009)

There are NO minimal translation surfaces in $\mathbb{H}^3$ of type 1. The only minimal translation surfaces in $\mathbb{H}^3$ of type 2 are totally geodesic planes.
The Heisenberg group $\text{Nil}_3 \sim \mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2} (x_1 y_2 - x_2 y_1))$$

$$g = dx^2 + dy^2 + \left[ dz + \frac{1}{2} (ydx - xdy) \right]^2$$
Heisenberg group $\text{Nil}_3 \sim \mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := \left( x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2} (x_1 y_2 - x_2 y_1) \right)$$

$$g = dx^2 + dy^2 + [dz + \frac{1}{2} (ydx - xdy)]^2$$

rich properties: homogeneous space, the group of isometries has dimension 4, contact Riemannian structure
Heisenberg group $Nil_3 \sim \mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2} (x_1 y_2 - x_2 y_1))$$

$$g = dx^2 + dy^2 + [dz + \frac{1}{2} (y dx - x dy)]^2$$

rich properties: homogeneous space, the group of isometries has dimension 4, contact Riemannian structure

Lie algebra of $Iso(Nil_3)$ is generated by Killing v. f.

$$E_1 = \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial z}$$
$$E_2 = \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z}$$
$$E_3 = \frac{\partial}{\partial z}$$
$$E_4 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$
$\mathbb{Nil}_3$

- $E_4$ generates the group of rotations around $z$-axis $\sim SO(2)$
Translation surfaces in the Heisenberg group $\text{Nil}_3$

$\text{Nil}_3$

- $E_4$ generates the group of rotations around $z$-axis $\sim SO(2)$
- $G_1 = \{(t, 0, 0) | t \in \mathbb{R}\}$, $G_2 = \{(0, t, 0) | t \in \mathbb{R}\}$, $G_3 = \{(0, 0, t) | t \in \mathbb{R}\}$
A surface in $Nil_3$ is translation invariant if it is invariant under the action of 1-parameter subgroup generated by a Killing vector field of the form $a_1 E_1 + a_2 E_2 + a_3 E_3$, $a_1^2 + a_2^2 + a_3^2 \neq 0$. 
Translation surfaces in the Heisenberg group $Nil_3$

- $E_4$ generates the group of rotations around $z$-axis $\sim SO(2)$
- $G_1 = \{(t, 0, 0)| t \in \mathbb{R}\}, \ G_2 = \{(0, t, 0)| t \in \mathbb{R}\}, \ G_3 = \{(0, 0, t)| t \in \mathbb{R}\}$

**Definition (Figueroa, Mercuri, Pedrosa - 1999)**

A surface in $Nil_3$ is **translation invariant** if it is invariant under the action of 1-parameter subgroup generated by a Killing vector field of the form $a_1 E_1 + a_2 E_2 + a_3 E_3$, $a_1^2 + a_2^2 + a_3^2 \neq 0$.

**Proposition (Figueroa, Mercuri, Pedrosa - 1999)**

Let $M$ in $Nil_3$ be invariant under the 1-parameter group generated by

$$a_1 E_1 + a_2 E_2 + a_3 E_3, \ a_1^2 + a_2^2 \neq 0.$$ 

Then it is equivalent to a surface invariant under $G_1$. 
Flat translation invariant surfaces

translation invariant surfaces : restrict to $G_1$ and $G_3$
Flat translation invariant surfaces

translation invariant surfaces: restrict to $G_1$ and $G_3$

**Proposition (Inoguchi - 2005)**

Let $M$ be a surface invariant under $G_3 = \{(0, 0, t) : t \in \mathbb{R}\}$. Then $M$ is locally expressed as

$$(0, 0, v) \cdot (x(u), y(u), 0) \quad , \quad u \in I, v \in \mathbb{R}.$$ 

$I$ - open interval, $u$ - arclength parameter
Translation surfaces in the Heisenberg group $Nil_3$

Flat translation invariant surfaces

Translation invariant surfaces: restrict to $G_1$ and $G_3$

**Proposition (Inoguchi - 2005)**

Let $M$ be a surface invariant under $G_3 = \{(0, 0, t) : t \in \mathbb{R}\}$. Then $M$ is locally expressed as

$$(0, 0, v) \cdot (x(u), y(u), 0), \quad u \in I, v \in \mathbb{R}.$$ 

$I$ - open interval, $u$ - arclength parameter

**Remark 1.** $(x, y, 0)$ and $(0, 0, v)$ commute.

**Remark 2.** $M$ is flat
Flat translation invariant surfaces

Proposition (Inoguchi - 2005)

Let $M$ be a surface invariant under $G_1 = \{(t, 0, 0), \ t \in \mathbb{R}\}$. Then $M$ is flat if and only if it is locally equivalent to the graph of

$$f(x, y) = \frac{xy}{2} + \frac{1}{2A} \left[ y\sqrt{y^2 - A^2} - A^2 \log|y + \sqrt{y^2 - A^2}| \right], \quad A \in \mathbb{R}^*.$$

Proof.

idea: the translation invariant surface $(G_1)$ is locally parametrized as the graph

$$(x, 0, 0) \cdot (0, y, v(y)) = \left(x, y, v(y) + \frac{xy}{2}\right).$$

compute $K$ + solve ODE
Minimal $G_1$ - invariant surfaces

Proposition (Inoguchi - 2005)

Let $M$ be a surfaces invariant under $G_1 = \{(t, 0, 0), \ t \in \mathbb{R}\}$. Then $M$ is minimal if and only if it is locally equivalent to the graph of

$$f(x, y) = \frac{xy}{2} + a \left[ y\sqrt{1+y^2} + \log(y + \sqrt{1+y^2}) \right], \quad a \in \mathbb{R}^*.$$
Minimal $G_1$ - invariant surfaces

Proposition (Inoguchi - 2005)

Let $M$ be a surfaces invariant under $G_1 = \{(t, 0, 0), \ t \in \mathbb{R}\}$. Then $M$ is minimal if and only if it is locally equivalent to the graph of

$$f(x, y) = \frac{xy}{2} + a \left[ y \sqrt{1 + y^2} + \log(y + \sqrt{1 + y^2}) \right], \quad a \in \mathbb{R}^*.$$ 

Extensions: using translation to the right for curves in the $xz$-plane and $yz$-plane: no flatness results.
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Why nothing about $G_4$?

$G_4$ invariant surfaces are nothing but rotational surfaces around $z$-axis ($G_4 = SO(2)$)

Classification results: Caddeo, Piu, Ratto - 1996
"Sum" of two curves

work in progress with Rafael López

$S^3$ hypersurface in $\mathbb{R}^4 \equiv \mathbb{H}$ (noncommutative field of quaternions)
$S^3$ group of unit quaternions

$\alpha(s), \beta(t)$ curves on $S^3$ (parametrized by arclength)
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Example (well known)

$$r(s, t) = (\cos s \cos t, \sin s \cos t, \cos s \sin t, \sin s \sin t).$$

• $\alpha = (\cos s, \sin s, 0, 0), \beta(t) = (\cos t, 0, \sin t, 0)$: translation surface

• minimal and II-minimal
Generalities

From now on $\alpha(s) = (\cos s, \sin s, 0, 0)$. 

$\beta(t) \in S^3$: $\exists q = q(t) \in S^2 \subset \text{Im} H$ s.t.

$$\beta'(t) = \beta(t) \cdot q(t) \quad g = ds^2 + 2Fdsdt.$$ 

$F = \langle ir, rq \rangle$

$N = j \cdot \zeta r, \zeta \in S^1 \subset C$ 

$ad(r)(q) = xi \pm \sqrt{1-x^2}ij \cdot \zeta.$ 

The function $x$ does not depend on $s$. 

Marian Ioan MUNTEANU (UAIC)  
On the geometry of translation surfaces  
Varna, June 2009  
34 / 39
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$$\langle \text{ad}(r)(q), j\zeta \rangle = 0$$
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there exists $x \in (0, 1)$ depending on $s$ and $t$ such that

$$N = \pm \frac{1}{\sqrt{1-x^2}} (xr + irq)$$

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From now on \( \alpha(s) = (\cos s, \sin s, 0, 0) \).

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The function \( x \) does not depend on \( s \)!!
First results

Proposition (López, M. - 2009)

The surface $S$ is flat.
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Example (the easiest: $q' = 0$)

$$\beta(t) = (\cos t, \sin t \sin \theta_0, \sin t \cos \theta_0 \cos \psi_0, \sin t \cos \theta_0 \sin \theta_0).$$

Proof.

$$\frac{\partial}{\partial t} \text{ad}(r)(q) = \text{ad}(r)(q') \quad \beta'(t) = \xi_0 \beta(t)$$

$$\xi_0 = \sin \theta_0 \ i + jw_0, \quad w_0 \in \mathbb{C}, \ |w_0| = \cos \theta_0, \ \theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
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Remark. All these surfaces are minimal.
Other results

Recall $N = j\zeta r$, $\zeta \in S^1 \subset \mathbb{C}$

$$\zeta = \cos \varphi + \sin \varphi i, \quad \varphi = \varphi(s, t)$$

Weingarten operator: $A = \begin{pmatrix}
\frac{-x}{\sqrt{1 - x^2}} & \frac{1 + x\varphi_t}{\sqrt{1 - x^2}} \\
\frac{1}{\sqrt{1 - x^2}} & \frac{x + \varphi_t}{\sqrt{1 - x^2}}
\end{pmatrix}$
Other results

Recall $N = j\zeta r$, $\zeta \in S^1 \subset \mathbb{C}$

$\zeta = \cos \varphi + \sin \varphi \, i$, $\varphi = \varphi(s, t)$

Weingarten operator: $A = \begin{pmatrix} -\frac{x}{\sqrt{1 - x^2}} & \frac{1 + x\varphi_t}{\sqrt{1 - x^2}} \\ 1 & -\frac{x + \varphi_t}{\sqrt{1 - x^2}} \end{pmatrix}$

Proposition (López, M. - 2009)

The surface $S$ cannot be totally geodesic in $S^3$. 
Minimality

Proposition (López, M. - 2009)

The surface $S$ is minimal if and only if $\varphi(s, t) = -2 \left(s + \int x(t) dt\right)$. Moreover

$$\text{ad}(r)(q) = x \mathbf{i} - \sqrt{1 - x^2} \left(-\sin \left(2 \int x(t) dt + 2s\right) \mathbf{j} + \cos \left(2 \int x(t) dt + 2s\right) \mathbf{k}\right)$$

where $x = x(t)$ is a smooth function.
Minimality

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The surface \( S \) is minimal if and only if \( \varphi(s, t) = -2 \left(s + \int x(t) dt\right) \).

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\]

where \( x = x(t) \) is a smooth function.

**Difficulties:** In order to give an explicit expression for \( \beta \) we have to solve the following QODE

\[
\beta'(t) = \mu(t)\beta(t) \quad , \quad \mu(t) \text{ is known}
\]
Problem

Find a 3-dimensional space and an embedding such that the following object becomes $\|\|\$-minimal or $\|\|\$-flat
Ceramic joke

Find a 3-dimensional space and an embedding such that the following object becomes $\|\cdot\|$-minimal or $\|\cdot\|$-flat
THANK YOU

FOR

ATTENTION !