Biharmonic submanifolds of $\mathbb{S}^4$

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Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of E:
harmonic maps

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Critical points of E: harmonic maps

Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0 \end{aligned}$$

Critical points of E_2: biharmonic maps

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- is a fourth-order non-linear elliptic equation;
- any harmonic map is biharmonic;
- a non-harmonic biharmonic map is called proper biharmonic;
- the biharmonic submanifolds M of a given space N are the submanifolds such that the inclusion map $i : M \to N$ is biharmonic. (the inclusion map $i : M \to N$ is harmonic if and only if M is minimal)

Biharmonic submanifolds in the Euclidean space

$$R^N = 0 \Rightarrow \tau_2(\varphi) = -\Delta^\varphi \tau(\varphi)$$

Definition (Chen)

A submanifold $i : M \longrightarrow \mathbb{R}^n$ is biharmonic if it has harmonic mean curvature vector field, i.e.

$$\Delta^i H = 0 \Leftrightarrow \Delta^i \tau(i) = 0.$$

Non existence of proper biharmonic submanifolds

For any of the following classes of submanifolds biharmonicity is equivalent to minimality:

- submanifolds of $\mathbb{E}^3(c)$, $c \leq 0$ (Chen/Caddeo - Montaldo - O.)
- curves of $\mathbb{E}^n(c)$, $c \leq 0$ (Dimitric/Caddeo - Montaldo - O.)
- submanifolds of finite type in $\mathbb{R}^n$ (Dimitric)
- hypersurfaces of $\mathbb{R}^n$ with at most two principal curvatures (Dimitric)
- pseudo-umbilical submanifolds of $\mathbb{E}^n(c)$, $c \leq 0$ with dimension $m \neq 4$ (Dimitric/Caddeo - Montaldo - O.)
- hypersurfaces of $\mathbb{R}^4$ (Hasanis - Vlachos)
- spherical submanifolds of $\mathbb{R}^n$ (Chen)

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- spherical submanifolds of $\mathbb{R}^n$ (Chen)

It is still open the following

Generalized Chen's Conjecture

Biharmonic submanifolds of $\mathbb{E}^n(c)$, $n > 3$, $c \leq 0$, are minimal.

Biharmonic curves in spheres

Biharmonic curves of $\mathbb{S}^2$ (Caddeo - Montaldo - Piu, 2001)

An arc length parameterized curve $\gamma : I \to \mathbb{S}^2$ is proper biharmonic if and only if it is the circle of radius $\frac{1}{\sqrt{2}}$.

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Biharmonic curves of $\mathbb{S}^3$ (Caddeo - Montaldo - O., 2001)

An arc length parameterized curve $\gamma : I \to \mathbb{S}^3$ is proper biharmonic if and only if it is either the circle of radius $\frac{1}{\sqrt{2}}$, or a geodesic of the minimal Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ with slope different from ± 1.

Biharmonic curves of $\mathbb{S}^n$, $(n \geq 3)$ (Fetcu - O., 2009)

An arc length parameterized curve $\gamma : I \to \mathbb{S}^n$ is proper biharmonic if and only if it is either the circle

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(\sqrt{2}s)e_1 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}s)e_2 + \frac{1}{\sqrt{2}}e_3,$$

or a helix

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4,$$

where $A = \sqrt{1+k_1}$, $B = \sqrt{1-k_1}$, $k_1 \in (0,1)$, and $\{e_i\}$ are constant unit vectors orthogonal to each other.

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Biharmonic curves of $\mathbb{S}^4$

Up to a totally geodesic embedding, the proper biharmonic curves of $\mathbb{S}^4$ are those of $\mathbb{S}^3$.

The biharmonic equation for submanifolds in spheres

Let $i : M^m \to \mathbb{S}^n$ be a submanifold. Then

$$\tau(i) = mH, \qquad \tau_2(i) = -m\Delta^i H + m^2 H,$$

thus i is biharmonic iff $\Delta^i H = mH$.

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(i) A submanifold $i: M^m \to \mathbb{S}^n$ is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) - mH = 0, \\ \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + m \operatorname{grad}(|H|^2) = 0. \end{cases}$$

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(ii) If M is a hypersurface of $\mathbb{S}^{m+1}$, then M is biharmonic if and only if

$$\begin{cases} \Delta^\perp H - (m - |A|^2) H = 0, \\ \\ 2A(\operatorname{grad}(|H|)) + m|H| \operatorname{grad}(|H|) = 0. \end{cases}$$

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Jiang, 1986/ Caddeo - Montaldo - O., 2002)

The composition property

$$\mathbb{S}^{n-1}(a) \xrightarrow{\text{biharmonic}} \mathbb{S}^n \qquad \Longleftrightarrow \qquad a = \frac{1}{\sqrt{2}}$$

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Properties

- M has parallel mean curvature vector field, and $|H| = 1$.
- M is pseudo-umbilical in $\mathbb{S}^n$, i.e. $A_H = |H|^2 \operatorname{Id}$.

Main examples of biharmonic submanifolds in $\mathbb{S}^n$

The product composition property

$$\mathbb{S}^{n_1}(a) \times \mathbb{S}^{n_2}(b) \xrightarrow{\text{biharmonic}} \mathbb{S}^n \quad \Longleftrightarrow \quad a = b = \frac{1}{\sqrt{2}} \quad \text{and} \quad n_1 \neq n_2$$

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Properties

- $M_1\times M_2$ has parallel mean curvature vector field, and $|H|\in(0,1)$.
- $M_1\times M_2$ is not pseudo-umbilical in $\mathbb{S}^n$.

The type of biharmonic submanifolds in spheres

Definition

A compact submanifold M of $\mathbb{S}^n$ is called of ℓ- type if the spectral decomposition of $\phi : M \to \mathbb{R}^{n+1}$ has exactly ℓ- terms, except for its center of mass, i.e. $\phi = \phi_0 + \sum_{j=1}^{\ell} \phi_j, \qquad \Delta\phi_j = \lambda_{t_j}\phi_j.$

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Theorem (Balmuş-Montaldo-O., 2007)

Let M^m be a compact constant mean curvature, $|H|^2 = k$, submanifold in $\mathbb{S}^n$. Then M is proper biharmonic if and only if either

(i) $|H|^2 = 1$ *and M is a 1- type submanifold with eigenvalue $\lambda = 2m$,*

or

(ii) $|H|^2 = k \in (0,1)$ *and M is a 2- type submanifold with the eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.*

Classification results (Balmuş-Montaldo-O., 2009)

For the classification of all the biharmonic submanifolds of $\mathbb{S}^n$ the strategy is:

- for hypersurfaces:
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- for hypersurfaces:
  divide the study according to the number $\mathbf{k}$ of distinct principal curvatures (which are functions)
- for submanifolds of higher codimension:
  impose geometric conditions on the mean curvature vector field.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\mathbf{k}=1$

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M^m is umbilical and proper biharmonic in $\mathbb{S}^{m+1}$

$$\Updownarrow$$

M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$

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Theorem

A biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$ has constant mean curvature.

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Theorem

Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

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Biharmonic surfaces of $\mathbb{S}^3$

A surface M^2 is proper biharmonic in $\mathbb{S}^3$ if and only if it is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$.

Biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ with $\mathbf{k} = 3$

Theorem

There exist no compact proper biharmonic hypersurfaces in the unit Euclidean sphere of constant mean curvature and with three distinct principal curvatures everywhere.

Theorem

A biharmonic hypersurface in $\mathbb{S}^4$ has constant mean curvature.

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Biharmonic hypersurfaces of $\mathbb{S}^4$

The only proper biharmonic compact hypersurfaces in $\mathbb{S}^4$ are the hypersphere $\mathbb{S}^3(\frac{1}{\sqrt{2}})$ and the torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.

Biharmonic surfaces of $\mathbb{S}^4$

Theorem

Let M^2 be a pseudo-umbilical surface of $\mathbb{S}^4$. Then M is proper biharmonic if and only if it is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$.

Theorem

Let M^2 be a surface with parallel mean curvature vector field in $\mathbb{S}^4$. Then M^2 is proper biharmonic in $\mathbb{S}^4$ if and only if it is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$.

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Theorem (Balmuş – O., 2009)

Let M^2 be a constant mean curvature surface in $\mathbb{S}^4$. Then M^2 is proper biharmonic in $\mathbb{S}^4$ if and only if it is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$.

Sketch of proof

We shall prove that $\nabla^{\perp} H = 0$.
Assume that $\nabla^{\perp} H \neq 0$.
Consider $\{E_1, E_2\}$ tangent to M, $\{E_3 = \frac{H}{|H|}, E_4\}$ normal to M.
Using the connection 1-forms w.r.t. $\{E_1, E_2, E_3, E_4\}$ and the tangent part of the biharmonic equation, we get $A_4 = 0$.
Case I. $A_3 = |H| \mathrm{Id} \Rightarrow M$ minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}}) \Rightarrow \nabla^{\perp} H = 0$ – contradiction.
Case II. $A_3 \neq |H| \mathrm{Id}$ + Gauss + Codazzi – contradiction.

Further studies

Conjecture

The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of generalized Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Conjecture

Any biharmonic submanifold in $\mathbb{S}^n$ has constant mean curvature.

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