SYMPLECTIC AND CONTACT GEOMETRY AS AN APRIORIC FOUNDATION OF STATISTICAL AND QUANTUM MECHANICS

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Some phase-space preliminaries.
(Just to remind (to) you).

(P, γ) – symplectic manifold:

P – a differential manifold

γ – a closed non-degenerate two-form on P:

$$d\gamma = 0 \qquad \gamma_{ab,c} + \gamma_{bc,a} + \gamma_{ca,b} = 0$$

$$(X \lrcorner \gamma = 0) \Rightarrow X = 0, \qquad \det\left[\gamma_{ab}\right] \neq 0.$$

Therefore, P – even-dimensional,

$$\dim P = 2n, \qquad n \in \mathbb{N}.$$

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The reciprocal bivector: γ^{-1}, $[\gamma^{ab}]$,

$$\gamma^{ac}\gamma_{cb} = \delta^a{}_b .$$

$$\gamma = \tfrac{1}{2}\gamma_{ab}\, dz^a \wedge dz^b , \qquad \gamma^{-1} = \tfrac{1}{2}\gamma^{ab}\frac{\partial}{\partial z^a}\wedge\frac{\partial}{\partial z^b} ,$$

z^a, $a = 1, \dots, 2n$ - coordinates on P,

$$\gamma_{ab} = -\gamma_{ba} , \qquad \gamma^{ab} = -\gamma^{ba}$$

Darboux coordinates:

$$[\gamma_{ab}] = \left[\begin{array}{c|c} 0_n & -I_n \\ \hline I_n & 0_n \end{array}\right] \qquad [\gamma^{ab}] = \left[\begin{array}{c|c} 0_n & I_n \\ \hline -I_n & 0_n \end{array}\right]$$

(A) (2)

Typical example: the cotangent bundle,

$$\left(T^*Q,\ d\Theta_Q\right),$$

Θ_Q - Cartan one-form,

Q-induced coordinates (q^i, p_i),

$$\Theta_Q = p_i\, dq^i, \qquad \omega_Q = d\Theta_Q = dp_i \wedge dq^i$$

(q^i, p_i) - automatically Darboux coordinates.

Canonical transformations: diffeomorphisms $\varphi : P \to P$ onto, preserving ω:

$$\varphi^* \cdot \omega = \omega$$

Infinitesimal ones - vector fields X on P generating one-parameter groups of canonical transformations:

$$\mathcal{L}_X \gamma = 0$$

$$d(X \lrcorner \gamma) = 0$$

Locally, in simply-connected P, globally:

$$X \lrcorner \gamma = dF \quad , \quad \text{denoted } X_F$$

$$X_F = \frac{\partial F}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial}{\partial p_i}$$

F - infinitesimal generator, i.e., Hamiltonian, of the Hamiltonian vector field X_F. Hamilton equations of integral curves:

$$\frac{dq^i}{dt} = \frac{\partial F}{\partial p_i} \qquad \frac{dp_i}{dt} = - \frac{\partial F}{\partial q^i}$$

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Poisson brackets: $F, G \in C^1(P)$

$$\{F, G\} := \gamma^{ab} F_{,a} G_{,b} .$$

in Darboux coordinates:

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$$

$C^\infty(P)$ - Lie algebra under $\{\ ,\ \}$:

(i) $\mathbb{R}$ - bilinear

(ii) skew-symmetric

(iii) Jacobi identity

Submanifolds of (P, γ), classification. A 4

Quite different than in Riemann spaces (skew-symmetry of the „metric" γ.

$M \subset P$ – submanifold of co-dimension m; $\dim M = 2n - m$.

$T_pM \subset T_pP$ – tangent space to M at $p \in M$.

$T_pM^\perp \subset T_pP$ – symplectic dual of T_pM:

$$T_pM^\perp := \{x \in T_pP : x \lrcorner \gamma_p = 0\} .$$

– „symplectic-orthogonal" to M at $p \in M$.

$\dim T_pM^\perp = m$

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$K_pM := T_pM^{\perp} \cap T_pM$ - internal singularity at p.

denote: $k := \dim K_pM$

$Cl\, M = (k, m-k)$

$(m, 0)$ - $Cl\ \underline{I}$

$(0, m)$ - $Cl\ \underline{II}$

$(m-k)$ - always even

$P'_pM := T_pM / K_pM$ - always symplectic, γ'_p - „projection" of γ_p to P'_pM.

$$\dim P'_pM = 2n - (m-k) = 2\left(n - \frac{m-k}{2}\right)$$

$M \ni p \mapsto K_pM \subset T_pM$ - INTEGRABLE DISTRIBUTION (because $d\gamma = 0$)

$K(M)$ - integral foliation, $(P'(M), \gamma') = (M/K(M), \gamma')$ - reduced phase space.

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Analytical description:

$$M := \{ p \in P : F_r(p) = 0 ,\ r = 1, \dots, m \},$$

F_r - functionally independent in some neighbourhood of M.

$$\mathrm{Rank}\left[\frac{\partial F_r}{\partial z^b}\right] = \mathrm{Rank}\left[\frac{\partial F_r}{\partial q^i}, \frac{\partial F_r}{\partial p_i}\right] = m$$

$$m - k = \mathrm{Rank}\left[\{F_r, F_s\}\right]\Big|_M$$

Extreme special cases:

(i) Co-isotropic M: $\mathrm{Cl}\, M = (m, 0) = \mathrm{I}$ - first class

$$\{F_r, F_s\}\big|_M = 0, \text{ i.e., } \{F_r, F_s\} = C^z{}_{rs} F_z ,\quad C^z{}_{rs} \text{ - smooth}$$

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If foliation: $F_r = \text{const}$,

$$\{F_r, F_s\} = 0 \quad \text{– Poisson-commuting}$$

The lowest possible dimension: $2n - m = n$; $m = n$

$$T_p M^{\perp} \subset T_p M$$

If $m = 1$, it is always so !!!

(ii) Isotropic M:

$\gamma \| M = 0$; $\gamma_p(x, y) = 0$ for any $p \in M$, $x, y \in T_p M$

The highest possible dimension:

$2n - m = n$; $m = n$

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(iii) Lagrangian submanifolds :

co-isotropic $\wedge$ isotropic , $m = n$

maximally-dimensional isotropic, minimally-dimensio-nal co-isotropic ,

$$\{F_r, F_s\}|M = 0 \;, \quad r,s = 1,\dots,n,$$

i.e. $\{F_r, F_s\} = C^z{}_{rs} F_z$, $C^z{}_{rs}$ - smooth.

If foliation, $F_r = a_r = \mathrm{const}$, $r = 1,\dots,n$,

then $\{F_r, F_s\} = 0$

A MAXIMAL system of Poisson-commuting functions

If $P = T^*Q$ and M_a - transversal to the fibers T_q^*Q, then:

$$p_i = \frac{\partial S(q,a)}{\partial a^i}, \quad i = 1, \ldots, n.$$

$$\det \left[\frac{\partial^2 S}{\partial q^i \partial a^j} \right] \neq 0$$

And any single Q-transversal isotropic may be represented in this potential way:

$$p_i = \frac{\partial S(q)}{\partial q^i}, \quad i = 1, \ldots, n$$

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Theorem: (A) (7)

If M is co-isotropic, m is Lagrangian and $m \subset M$, then m is foliated by singular fibres, i.e., fibres of $K(M)$. If M is not co-isotropic, there are no Lagrangian manifolds m in M.

What is and what does mean Hamilton-Jacobi equation?

t - time, q^i - generalized coordinates, H - Hamiltonian

$$\frac{\partial S}{\partial t} + H\left(t, q^a, \frac{\partial S}{\partial q^a}\right) = 0$$

Stationary, when H time-independent

$$S = -Et + S_0(q^a), \quad E \text{ - constant (energy).}$$

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$$H\left(q^a, \frac{\partial S}{\partial q^a}\right) = E$$

Homogeneous formulation:

$$\Omega\left(x^\mu, \frac{\partial S}{\partial x^\mu}\right) = 0$$

Ω - „energy function" (J. L. Synge)

What geometrically do these equations mean?

In cotangent bundles over the configuration „space-times" parametrized respectively by (t, q^a) or (x^μ), we are given constraints submanifolds $\mathcal{M}$ described by equations:

$$p_t + H(t, q^a, p_a) = 0$$

$$\Omega(x^\mu, p_\mu) = 0,$$

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p_t denoting „canonical momentum" conjugate to the „time coordinate" t. In stationary formalism when the energy value E is fixed we are given constraints submanifold given by equation

$$H(q^a, p_a) - E = 0 .$$

They are odd-dimensional submanifolds of codimension one and realistic motions are given by the corresponding one-dimensional fibres of $K(\mathcal{M})$, i.e., integral curves of vector fields:

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial t}\frac{\partial}{\partial p_t} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}$$

$$X_\Omega = \frac{\partial \Omega}{\partial p_r}\frac{\partial}{\partial x^r} - \frac{\partial \Omega}{\partial x^r}\frac{\partial}{\partial p_r}$$

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and in the stationary case

$$X_{H,E} = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}\ .$$

Ordinary differential equations for their integral curves, e.g.,

$$\frac{dx^\mu}{d\tau} = \frac{\partial \Omega}{\partial p_\mu}\ , \qquad \frac{dp_\mu}{d\tau} = -\frac{\partial \Omega}{\partial x^\mu}$$

$$\frac{dq^a}{dt} = \frac{\partial H}{\partial p_a}\ , \qquad \frac{dp_a}{dt} = -\frac{\partial H}{\partial q^a}$$

must be combined with the constraints equations (their integral curves are tangent to constraints.

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They are characteristic bands of the Hamilton-Jacobi equations and their projections onto configuration space-times or configuration spaces are characteristics, i.e., rays. This is just the optical-mechanical analogy; Hamilton-Jacobi equations being mechanical eikonal equations of geometric waves, rays being organized into coherent families (Synge, Dirac) associated with solutions S for geometric „waves".

Hamilton-Jacobi equation, e.g.,

$$\Omega\left(x^{\mu}, \frac{\partial S}{\partial x^{\mu}}\right) = 0$$

does mean that the Lagrange manifold

$$\mathcal{M}_S \; : \quad p_{\mu} = \frac{\partial S}{\partial x^{\mu}}$$

is placed on constraints $\mathcal{M}$:

$$\Omega(x^\mu, p_\mu) = 0$$

What is the complete integral of the Hamilton-Jacobi equation? The family of solutions:

$$S(t, q^i; \alpha^i) \qquad i = 1, \dots, n$$

$$S(x^\mu, \alpha^i) \qquad \mu = 1, \dots, n+1 \quad i = 1, \dots, n$$

where:

- for any α, $S(\cdot, \cdot, \alpha)$ is a solution of the Hamilton-Jacobi equation
- $\det\left[\frac{\partial^2 S}{\partial q^i \partial \alpha_j}\right] \neq 0$
- $\mathrm{Rank}\left[\frac{\partial^2 S}{\partial x^\mu \partial \alpha_j}\right] = n.$

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This means that dynamical constraints $\mathcal{M}$ are foliated by some family of Lagrangian submanifolds

$$\mathcal{M}_{S(\cdot,\alpha)} .$$

Some strange fact (Schiller, Van Vleck):

Take:

$$D = \det\left[\frac{\partial^2 S}{\partial q^i \partial \alpha_j}\right]$$

Geometrically - scalar density of weight one in Q.

Hamilton-Jacobi equation implies that:

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial q^i}\left(D \, \frac{\partial H}{\partial p_i}\left(q^a, \frac{\partial S}{\partial q^a}\right)\right) = 0 .$$

This is continuity equation with the density D and the current (contravariant scalar density of weight one)

$$j^i = D \frac{\partial H}{\partial p_i}\left(q, \frac{\partial S}{\partial q}\right) = D v^i(q)$$

Strange; what does it mean physically? What a balance? Of what?

In homogeneous, let us say, „relativistic" formalism let us take:

$$\mathcal{V} = D^\mu\, dx^1 \wedge \ldots \wedge \underset{\substack{\uparrow \\ dx^\mu \text{ removed}}}{\mu} \wedge \ldots dx^{n+1} \otimes d\alpha^1 \wedge \ldots \wedge d\alpha^m$$

D^μ - the minor of $\left[\frac{\partial^2 S}{\partial x^\mu \partial \alpha^i}\right]$ obtained by removing the μ-th-column

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \;, \quad j^\mu = (-1)^\mu D^\mu \text{ - continuity equation}$$

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B 1

$\mathbb{R}^{2n}$ topology and geometry of the phase space

$\mathbb{R}^{n}$ topology and geometry of the configuration space

Kernel representation of operators:

$$(\hat{A}\Psi)(q) = \int A[q, q'] \, \Psi(q') \, d_n q'$$

Not always works with usual functions. Take, e.g., operators $\hat{1}$, $\hat{q}^a$, $\hat{p}_a$,, $(\hat{1}\Psi)(q) = \Psi(q)$

$$(\hat{q}^a \Psi)(q) = q^a \Psi(q)$$

$$(\hat{p}_a \Psi)(q) = \frac{\hbar}{i} \frac{\partial \Psi}{\partial q^a}$$

Their kernels $A(q, q')$ are respectively distributions:

$$\delta(q-q')$$

$$q^a \delta(q-q')$$

$$\frac{\hbar}{i} \frac{\partial}{\partial q^a} \delta(q-q')$$

Weyl-Wigner-Moyal prescription:

$$A(q,p) = \int A\left[q^i + \frac{\xi^i}{2}, q^i - \frac{\xi^i}{2}\right] e^{-\frac{i}{\hbar} p_k \xi^k} d\xi^1 \ldots d\xi^n$$

$$\langle q|\hat{A} q'\rangle = A[q,q'] = \frac{1}{(2\pi\hbar)^n} \int e^{-\frac{i}{\hbar} p_k (q^k - q^{k\prime})} A\left(\frac{q^l + q'^l}{2}, p_m\right) d_n p =$$

$$= \frac{1}{h^n} \int \exp\left(-\frac{i}{\hbar} p_k (q^k - q^{k\prime})\right) A\left(\frac{q^l+q'^l}{2}, p_m\right) d_n p$$

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$$(\hat{A}\,\hat{B}\,\psi)(q) = \int A[q,q'']\,B[q'',q']\,\psi(q')\,d_n q''\,d_n q'$$

$$(A[\,,\,]\perp B[\,,\,])(q,q') = \int A[q,q'']\,B[q'',q']\,d_n q''$$

$$(A[\,,\,]\perp B[\,,\,])[q,q'] = \frac{1}{(2\pi\hbar)^n}\int e^{-\frac{i}{\hbar}p_k(q^k-q^{k\prime})}(A*B)\left(\frac{q^l+q'^l}{2},p_m\right)d_n p$$

$$(A*B)(q,p) = \left(\frac{2}{2\pi\hbar}\right)^n\int \exp\left(\frac{2i}{\hbar}\left[(p_i''-p_i)(q'^i-q^i)-(p_i'-p_i)(q''^i-q^i)\right]\right)\times$$

$$\times\,A(q',p')\,B(q'',p'')\,d_n q'\,d_n p'\,d_n q''\,d_n p''$$

Geometrically, in symplectic terms:

$$(A * B)(z) = \left(\frac{2}{2\pi\hbar}\right)^{2n} \int \exp\left[-\frac{2i}{\hbar}\Gamma_{ab}(z^a-u^a)(z^b-v^b)\right] A(u)\, B(v)\, d_{2n}u\, d_{2n}v =$$

$$= 2^{2n} \int \exp\left[-\frac{2i}{\hbar}\Gamma_{ab}(z^a-u^a)(z^b-v^b)\right] A(u)\, B(v)\, d\Gamma(u)\, d\Gamma(v),$$

where: $[\Gamma_{ab}] = -[\Gamma^{ab}] = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ – the natural symplectic two-form on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^{n*}$, and $d\Gamma$ denotes the natural dimension-less volume element on the phase space,

$$d\Gamma = \left(\frac{1}{2\pi\hbar}\right)^n dq^1 \ldots dq^n\, dp_1 \ldots dp_n = \frac{1}{h^n} dq^1 \ldots dq^n\, dp_1 \ldots dp_n.$$

The above Weyl-Wigner-Moyal product, being isomorphic

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to the operator multiplication is associative, non-commutative, the constant function equal identically to one corresponds to the identity, operator, and complex conjugation corresponds to the hermition conjugation of operators. All formal relations are preserved, e.g.,

$$\overline{A * B} = \bar{B} * \bar{A},$$

and there are other obvious properties, e.g.,

$$\int A * B \, d\Gamma = \int AB \, d\Gamma.$$

although in general $A * B \neq AB$, and if both A, B depend only on the positions q, or on the momenta p, then $A * B = AB$. Let us quote other important formulas, e.g.,

$$q^i * A = q^i A - \frac{1}{2}\frac{\hbar}{i}\frac{\partial A}{\partial p_i}$$

$$p_i * A = p_i A + \frac{1}{2}\frac{\hbar}{i}\frac{\partial A}{\partial q^i} .$$

Quantum Poisson bracket, i.e., Moyal bracket, is given by:

$$\{A, B\}_{Quan} = \frac{1}{\hbar i}\left(A * B - B * A\right) .$$

Obviously, when $\hbar \to 0$ we have:

$$\lim_{h \to 0} A * B = A B$$

$$\lim_{h \to 0} \{A, B\}_{Quant} = \{A, B\} , \text{ the}$$

the classical Poisson bracket.

(B) (4)

All this may be done for the density operator $\hat{\varrho}$ as well, in particular for the pure state density operator,

$$\varrho[q, q'] = \langle q | \hat{\varrho} | q' \rangle = \Psi(q)\, \overline{\Psi}(q').$$

The corresponding ϱ is the well-known Wigner function:

$$\varrho(q, p) = \frac{1}{(2\pi)^n} \int \overline{\Psi}\left(q^i - \frac{\hbar \tau^i}{2}\right) e^{-i\tau^a p_a}\, \Psi\left(q^i + \frac{\hbar \tau^i}{2}\right) d_n\tau.$$

Except special situation like harmonic oscillator, it is not positive, although it is always positively definite in the quantum sense:

$$\int \varrho * (\overline{A} * A) = \int \varrho (\overline{A} * A) > 0$$

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for any physical quantity A.

Marginal distributions:

$$\int \rho(q,p)\,\frac{d_n p}{(2\pi\hbar)^n} = \overline{\psi}(q)\,\psi(q) \geq 0$$

$$\int \rho(q,p)\,d_n q = \overline{\varphi}(p)\,\varphi(p) \geq 0,$$

where φ – the momentum representation of the quantum state (Fourier distribution).

If ψ localized in positions or momenta:

$$\psi_x(q) = \delta_x(q) = \delta(q-x)$$

$$\psi_\pi(q) = \exp\left(\frac{i}{\hbar}\pi_a q^a\right), \quad \varphi_\pi(p) = \delta(p-\pi)$$

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we respectively obtain:

$$\varrho(q,p) = \delta(q^1 - x^1) \dots \delta(q^n - x^n) = \delta(q - x)$$

$$\varrho(q,p) = \delta(p_1 - \pi_1) \dots \delta(p_n - \pi_n) = \delta(p - \pi)$$

Distributions concentrated on Lagrangian manifolds!

In general - not, but let us introduce for the wave function

$$\Psi = \sqrt{D}\, \exp\left(\frac{i}{\hbar} S\right)$$

(D, S - $\hbar$-independent in the lowest order)

the following:

$$\varrho_{cl}(q,p) = D(q)\, \delta(p - \nabla S) = \overline{\Psi} \Psi(q)\, \delta\left(p_1 - \frac{\partial S}{\partial q^1}\right) \dots \delta\left(p_n - \frac{\partial S}{\partial q^n}\right)$$

If f – a linear function of q, p, in particular q^i, p_i, then for the expectation values we have:

$$\langle f \rangle_\varrho = \langle f \rangle_{\varrho_{cl}}$$

Generally not the case, but in the quasiclassical limit:

$$\lim_{\hbar \to 0} \varrho(q,p) = D(q)\, \delta\left(p_1 - \frac{\partial S}{\partial q^1}\right) \ldots \delta\left(p_n - \frac{\partial S}{\partial q^n}\right)$$

in the distribution sense. Probability distribution concentrated on the Lagrange manifold

$$m_S : \quad p_i = \frac{\partial S}{\partial q^i}$$

with weights $D(q) = \overline{\Psi}\, \Psi(q)$

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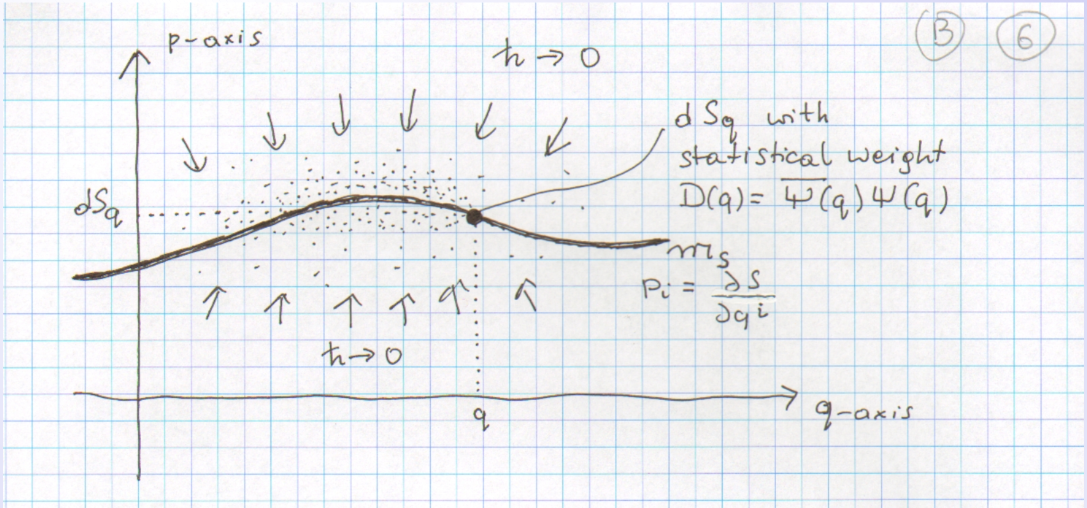

Literally positive Gauss distributions:

$$E_{(q_0,p_0)}(q,p) = \frac{1}{(\pi\hbar)^n} \exp\left(-\frac{1}{\hbar}\left[a_{lm}(q^l - q^l_0)(q^m - q^m_0) + a^{lm}(p_l - p_{l0})(p_m - p_{m0})\right]\right)$$

$$\int E_{(q_0,p_0)} = 1 , \qquad E_{(q_0,p_0)} * E_{(q_0,p_0)} = E_{(q_0,p_0)}$$

Pure states, oscillator.

$$\int \varrho \, E_{(q_0,p_0)} > 0 \quad \text{for any Wigner function}$$

$$\tilde{\varrho}(q,p) := \int E_{(q,p)}(\xi,\eta)\, \varrho(\xi,\eta)\, d_n\xi \, \frac{d_n\eta}{(2\pi\hbar)^n} \geqslant 0$$

Coarse-graining, averaging over $\hbar^n$-cells

HUSINI phase-space density. Positive, but bad marginal properties

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$$(\hat{A}\Psi)(q) = \frac{1}{(2\pi\hbar)^n}\int \exp\left(\frac{i}{\hbar}p_k(q^k - q^{k\prime})\right) A\left(\frac{q+q'}{2}, p\right)\Psi(q')\, d_n q'\, d_n p$$

$$\Psi(q) = f(q)\exp\left(\frac{i}{\hbar}S(q)\right) = \sqrt{D(q)}\;\exp\left(\frac{i}{\hbar}S(q)\right)$$

$$(\hat{A}\Psi)(q) \underset{\hbar\to 0}{\approx}$$

$$\underset{\hbar\to 0}{\approx} A\left(q^i, \frac{\partial S}{\partial q^i}\right) f(q)\exp\left(\frac{i}{\hbar}S(q)\right) + \frac{\hbar}{i}\left(\mathcal{L}_v f\right)\exp\left(\frac{i}{\hbar}S(q)\right) f(q)$$

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f – scalar density of weight one

v – velocity field („hydrodynamical")

$$v^i = \frac{\partial H}{\partial p_i}\left(q, \frac{\partial S}{\partial q}\right)$$

$\hat{A}\Psi = a\Psi$ – eigenequation

$\hbar \to 0$ lowest-order equations:

$$A\left(q, \frac{\partial S}{\partial q}\right) = a$$

$$\mathcal{L}_v f = \mathcal{L}_v \sqrt{D} = 0$$

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Dynamical level

$$\hbar i \frac{\partial \Psi}{\partial t} = \hat{H} \Psi$$

$\hbar \to 0$, short waves WKB -asymptotics:

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0$$

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial q^i} j^i = 0$$

$$j^i = D v^i_{H,S}(q,t) = D \frac{\partial H}{\partial p_i}\left(q, \frac{\partial S}{\partial q}\right),$$

i.e.,

$$\frac{\partial D}{\partial t} + \mathcal{L}_{v_{H,S}} D = 0$$

$$v^{i}_{H,S} = \frac{\partial H}{\partial p_i}\left(q, \frac{\partial S}{\partial q}\right)$$

For rigorously quantum Wigner functions ϱ there is no such local continuity equation.

$$\frac{\partial \varrho}{\partial t} = \frac{1}{\hbar i}\left(H * \varrho - \varrho * H\right)$$

for finite $\hbar$ one does not obtain the above local equations.

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Other asymptotic facts connected with the $\hbar \to 0$ asymptotic transition. Begin with scalar products

$$\psi_1 = \sqrt{D_1}\,\exp\left(\frac{i}{\hbar} S_1\right), \quad \psi_2 = \sqrt{D_2}\,\exp\left(\frac{i}{\hbar} S_2\right)$$

$$\langle \psi_1 | \psi_2 \rangle = \int \overline{\psi_1}(q)\, \psi_2(q)\, d_n q = D \exp\left(\frac{i}{\hbar}\varphi\right).$$

When $\hbar \to 0$ and D_1, S_1, D_2, S_2 are assumed to be $\hbar$-independent (at least in the lowest order of approximation), then:

$$\varphi = \mathrm{Stat}\,(S_2 - S_1)$$

where $\mathrm{Stat}\, f$ is the value of f at its stationary point,

$$\mathrm{Stat}\, f = f(x_0), \text{ where } df_{x_0} = 0.$$

Continuous superpositions:

$$\Psi_a = \sqrt{D_a}\, \exp\left(\frac{i}{\hbar} S_a\right)$$

Superposition coefficients:

$$t(a) = r(a)\, \exp\left(\frac{i}{\hbar} t_a\right)$$

$$\Psi = \int t(a)\, \Psi_a \, d_m a$$ – continuous superposition.

$$\Psi = \sqrt{D}\, \exp\left(\frac{i}{\hbar} S\right).$$

If $\hbar \to 0$, then

$$S(q) = \operatorname*{Stat}_a \left(t_a + S(q,a)\right)$$

***

The end of Part I.

Thank you for your attention!

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