On generalized Fourier transform for Kaup-Kuperschmidt type equations

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1. Introduction

- Kaup-Kuperschmidt equation

$$\partial_t f = \partial^5_{x^5} f + 10 f \partial^3_{x^3} f + 25 \partial_x f \partial^2_{x^2} f + 20 f^2 \partial_x f,$$

where $f \in C^\infty(\mathbb{R}^2)$.

- Lax representation

$$\begin{aligned} L &= \mathrm{i}\partial_x + q(x,t) - \lambda J, \\ M &= \mathrm{i}\partial_t + \sum_{k=0}^{5} V_k(x,t)\lambda^k, \end{aligned}$$

where

$$q = \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -u \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

u and f are interrelated by a Muira transformation as follows

$$f = -\mathrm{i}\partial_x u + \frac{1}{2}u^2.$$

The Lax pair is associated with the algebra $\mathfrak{sl}(3,\mathbb{C})$ with additional symmetries (reductions) imposed $\Rightarrow$ Caudrey-Beals-Coifman system.

Purpose of the talk: to demostrate how the generalized Fourier interpretation of the inverse scattering method for equations of the Kaup-Kuperschidt type can be achieved.

2. Some facts from the theory of solitons

- NEE and Lax pairs

$$\text{NEE} \qquad \Leftrightarrow \qquad [L(\lambda), M(\lambda)] = 0,$$

where

$$L(\lambda) = \mathrm{i}\partial_x + U(x,t,\lambda), \qquad U(x,t,\lambda) = q(x,t) - \lambda J,$$
$$M(\lambda) = \mathrm{i}\partial_t + V(x,t,\lambda), \qquad V(x,t,\lambda) = \sum_{k=0}^{N} V_k(x,t)\lambda^k,$$

All quantities $q(x,t)$ and $V_k(x,t)$ belong to a simple Lie algebra $\mathfrak{g}$ while $J \in \mathfrak{h}$ is real. We shall require that the potential q fulfills the condition

$$\lim_{x\to\pm\infty} |x|^l q(x,t) = 0, \qquad l \in \mathbb{Z}^+.$$

- Direct scattering problem
  - Auxiliary spectral problem (generalized Zakharov-Shabat system)
$$L\psi = (\mathrm{i}\partial_x + q(x,t) - \lambda J)\psi(x,t,\lambda) = 0.$$
    Then fundamental solutions ψ take values in G corr. to $\mathfrak{g}$.
  - Continuous spectrum of L: $\mathbb{R} \subset \mathbb{C}$.
  - Jost solutions
$$\lim_{x\to\pm\infty} \psi_{\pm}(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda J x} = \mathbb{1}.$$
  - Scattering matrix (data)
$$T(t,\lambda) = \hat{\psi}_+(x,t,\lambda)\psi_-(x,t,\lambda), \qquad \lambda \in \mathbb{R},$$
  - Dispersion law $\Rightarrow$ evolution of scattering data
$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0, \qquad f(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda)$$

– Fundamental analytic solutions

$$\chi^{\pm}(x,\lambda) = \psi_-(x,\lambda)S^{\pm}(\lambda) = \psi_+(x,\lambda)T^{\mp}(\lambda)D^{\pm},$$

where $S^{\pm}(\lambda)$, $T^{\pm}(\lambda)$ and $D^{\pm}(\lambda)$ are factors in the Gauss decomposition of $T(\lambda)$

$$T(\lambda) = \begin{cases} T^-(\lambda)D^+(\lambda)\hat{S}^+(\lambda), \\ T^+(\lambda)D^-(\lambda)\hat{S}^-(\lambda). \end{cases}$$

Hence we have (Riemman-Hilbert problem)

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G(\lambda), \qquad \lambda \in \mathbb{R},$$

$$G(\lambda) = \begin{cases} \hat{S}^-(\lambda)S^+(\lambda), \\ \hat{D}^-(\lambda)\hat{T}^+(\lambda)T^-(\lambda)D(\lambda). \end{cases}$$

- Algebraic reductions

  Let G_R be a discrete group acting on the set fundamental solutions $\{\psi(x,\lambda)\}$ as follows

$$\mathcal{K}[\psi(x,\kappa^{-1}(\lambda))] = \tilde{\psi}(x,\lambda).$$

The requirement of G_R-invariance of the lin. problem yields to certain symmetry conditions on U (and therefore on V).

Example 1 *Coxeter type reduction for* $\mathfrak{sl}(r+1,\mathbb{C})$

Impose the $\mathbb{Z}_{r+1}$ *reduction condition*

$$C[\psi(x,\kappa^{-1}(\lambda))]C^{-1} = \tilde{\psi}(x,\lambda)$$

where

$$\kappa : \lambda \to \omega\lambda, \qquad \omega = \mathrm{e}^{\frac{2\mathrm{i}\pi}{r+1}}, \qquad C = diag\,(1, \omega^r, \omega^{r-1} \ldots, \omega).$$

Thus the symmetry conditions for U and V read

$$\begin{aligned} CU(x,\omega^{-1}\lambda)C^{-1} &= U(x,\lambda) \qquad \Rightarrow \quad Cq(x)C^{-1} = q(x), \quad CJC^{-1} = \omega J, \\ CV(x,\omega^{-1}\lambda)C^{-1} &= V(x,\lambda) \qquad \Rightarrow \quad CV_k(x)C^{-1} = \omega^k V_k(x). \end{aligned}$$

Consequently $q(x)$ and J have the form

$$q = \sum_{k=1}^{r} q_k H_k, \qquad J = \sum_{\alpha\in\mathcal{A}} E_\alpha.$$

3. NEE of the Kaup-Kuperschmidt type

- Lax operators

$$
\begin{aligned}
L(\lambda) &= \mathrm{i}\partial_x + q(x,t) - \lambda J, \\
M(\lambda) &= \mathrm{i}\partial_t + \sum_{k=0}^{N} V_k(x,t)\lambda^k, \qquad N \neq 3l, \quad l \in \mathbb{Z}^+
\end{aligned}
$$

where q, J and V_k belong to $\mathfrak{sl}(3,\mathbb{C})$. Impose the additional $\mathbb{Z}_3$ Coxeter type reduction conditions

$$
\begin{aligned}
CqC^{-1} &= q, \quad CJC^{-1} = \omega J, \\
CV_kC^{-1} &= \omega^k V_k, \qquad \omega = \mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}, \quad C = \mathrm{diag}\,(1, \omega^2, \omega).
\end{aligned}
$$

Due to technical convenience we shall work in the following gauge

$$
J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
$$

$$q = \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -u \end{pmatrix} \mapsto q = \begin{pmatrix} 0 & cu & c^*u \\ c^*u & 0 & cu \\ cu & c^*u & 0 \end{pmatrix}, \qquad c = \frac{\omega - 1}{3}.$$

Therefore Coxeter's automorphism acts in $\mathfrak{sl}(3)$ by inner automorphism with the following matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Grading of the algebra $\mathfrak{sl}(3, \mathbb{C})$

  Since Coxeter's automorphism has a finite order $h = 3$ it determines a grading in $\mathfrak{sl}(3, \mathbb{C})$ as follows

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2, \qquad \mathfrak{g}^k = \{X \in \mathfrak{sl}(3); CXC^{-1} = \omega^k X\}.$$

  Obviously, the following equalities hold

$$q \in \mathfrak{g}^0, \qquad J \in \mathfrak{g}^1, \qquad V_k \in \mathfrak{g}^{k(mod(3))}.$$

- Spectral properties and direct scattering problem for $\mathbb{Z}_3$-reduced operator L

  - Continuous spectrum of L: consists of 6 rays l_a $(a = 1, \dots, 6)$ determined by

$$\operatorname{Im} \lambda\alpha(J) = 0.$$

  - Each ray l_a is connected with a $\mathfrak{sl}(2)$ subalgebra: $\{E_\alpha, E_{-\alpha}, H_\alpha\}$
  - The λ-plane is split into 6 sectors Ω_a and with each sector can be introduced different ordering of roots

$$\Delta_a^\pm = \{\alpha \in \Delta; \operatorname{Im}(\lambda\alpha(J)) \gtrless 0, \quad \forall\lambda \in \Omega_a\}.$$

  - Fundamental analytic solutions

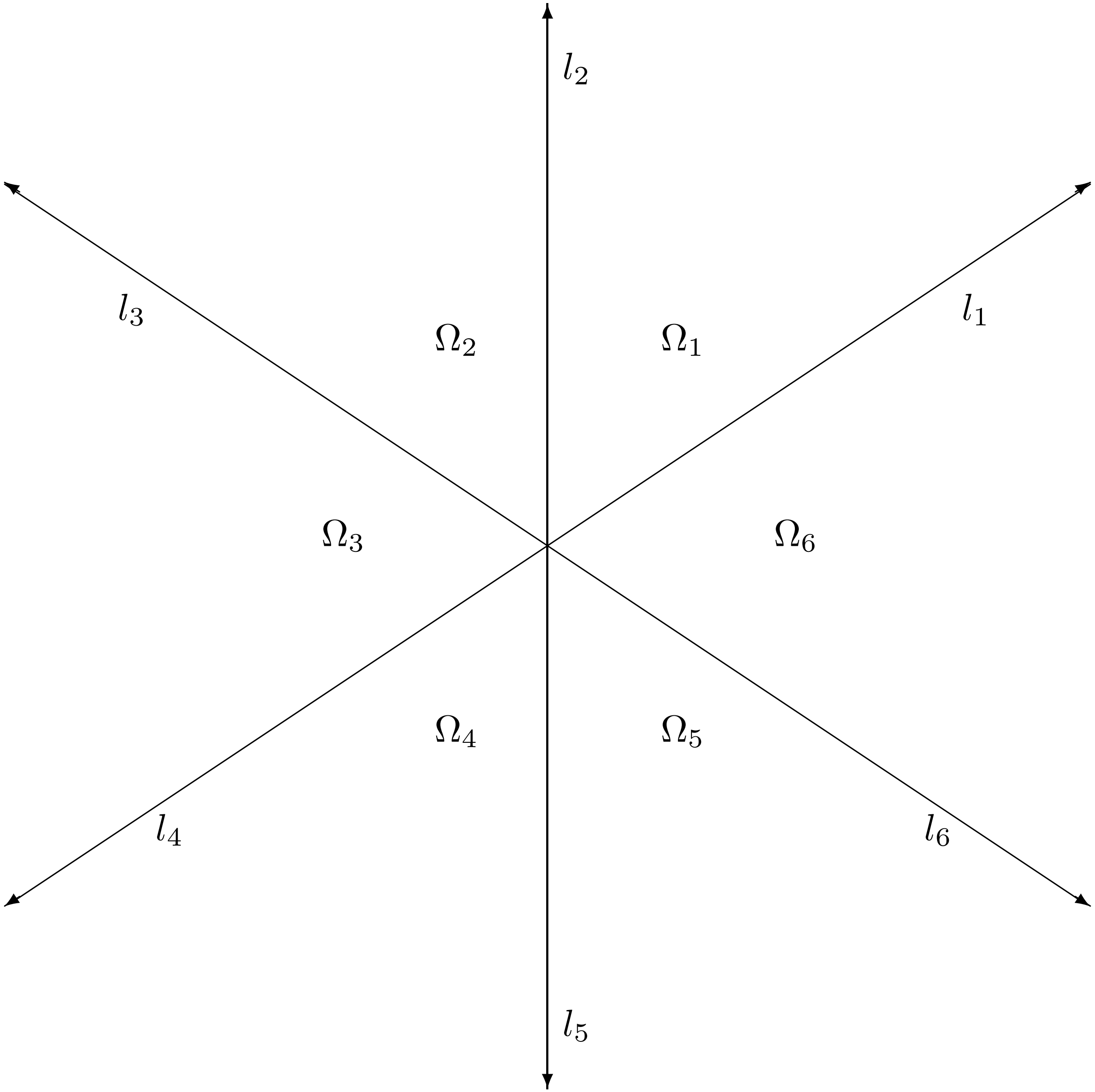
$$\chi^a(x,\lambda) = \chi^{a-1}(x,\lambda)G^a(\lambda), \qquad \lambda \in l_a.$$

$$G^a(\lambda) = \begin{cases} \hat{S}_a^-(\lambda)S_a^+(\lambda) \\ \hat{D}_a^-(\lambda)\hat{T}_a^+(\lambda)T_a^-(\lambda)D_a^+(\lambda), \end{cases}$$

where

$$S_a^{\pm}(\lambda) = \exp\left(\sum_{\beta\in\Delta_a^+} s_{a,\beta}^{\pm} E_{\pm\beta}\right), \qquad D_a^+ = \exp\left(\sum_{j=1}^{r} d_{a,j}^{+} H_j\right),$$

$$T_a^{\pm}(\lambda) = \exp\left(\sum_{\beta\in\Delta_a^+} t_{a,\beta}^{\pm} E_{\pm\beta}\right), \qquad D_a^- = \exp\left(\sum_{j=1}^{r} d_{a,j}^{-} w_0(H_j)\right).$$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4. Generalized Fourier transform for Kaup-Kuperschmidt type equations

- Squared solutions

$$\chi^a(x,\lambda) \to \begin{cases} e_\alpha^{(a)}(x,\lambda) = \pi\left[\chi^a(x,\lambda)E_\alpha(\chi^a(x,\lambda))^{-1}\right], \\ h_j^{(a)}(x,\lambda) = \pi\left[\chi^a(x,\lambda)H_j(\chi^a(x,\lambda))^{-1}\right], \end{cases}$$

where $\pi : \mathfrak{sl}(3) \to \mathfrak{sl}(3)/\ker \mathrm{ad}_J$.

- Recursion operator

Introduce the quantities

$$\mathscr{E}_\alpha^{(a)} = \chi^a E_\alpha \hat{\chi}^a = e_\alpha^{(a)} + d_\alpha^{(a)}, \qquad \mathscr{H}_j^{(a)} = \chi^a H_j \hat{\chi}^a = h_j^{(a)} + f_j^{(a)}.$$

to satisfy

$$\mathrm{i}\partial_x \mathscr{E}_\alpha^{(a)} + [q - \lambda J, \mathscr{E}_\alpha^{(a)}] = 0,$$
$$\mathrm{i}\partial_x \mathscr{H}_j^{(a)} + [q - \lambda J, \mathscr{H}_j^{(a)}] = 0.$$

After splitting the diagonal and off-diagonal part of above equations we get

$$\begin{aligned} i\partial_x e_\alpha + \pi[q,e_\alpha] + \pi[q,d_\alpha] &= \lambda\pi[J,e_\alpha], \\ i\partial_x d_\alpha + (\mathbb{1}-\pi)[q,e_\alpha] &= 0. \end{aligned}$$

Due to the existence of grading in $\mathfrak{sl}(3)$ the squared solutions have the representation

$$e_\alpha = e_{\alpha,0} + e_{\alpha,1} + e_{\alpha,2}, \qquad d_\alpha = \mathbf{d}_\alpha^1 J + \mathbf{d}_\alpha^2 J^2.$$

Substituting it into the above equations one gets

$$\begin{aligned} &\mathrm{i}\partial_x \mathbf{d}_\alpha^\sigma + \frac{1}{3}\mathrm{tr}\,\left([q,e_{\alpha,\sigma}]J^{3-\sigma}\right) = 0, \qquad \sigma = 1,2 \\ \Rightarrow\ &\mathbf{d}_\alpha^\sigma = \frac{\mathrm{i}}{3}\int_{\pm\infty}^x \mathrm{d}\,y\mathrm{tr}\,\left([q,e_\alpha]J^{3-\sigma}\right). \end{aligned}$$

On the other hand we have

$$\begin{aligned} &\mathrm{i}\partial_x e_{\alpha,0} + \pi[q,e_{\alpha,0}] = \lambda\pi[J,e_{\alpha,2}], \\ &\mathrm{i}\partial_x e_{\alpha,\sigma} + \frac{\mathrm{i}}{3}\pi[q,J^\sigma]\int_{\pm\infty}^x \mathrm{d}\,y\mathrm{tr}\,\left([q,e_{\alpha,\sigma}]J^{3-\sigma}\right) + \pi[q,e_{\alpha,\sigma}] = \lambda\pi[J,e_{\alpha,\sigma-1}]. \end{aligned}$$

As a result one obtains

$$\Lambda_0 e_{\alpha,0} = \lambda e_{\alpha,2}, \qquad \Lambda_\sigma e_{\alpha,\sigma} = \lambda e_{\alpha,\sigma-1},$$

where

$$\begin{aligned}
\Lambda_0 &= \mathrm{ad}_J^{-1}\left(\mathrm{i}\partial_x + \pi[q,.]\right),\\
\Lambda_\sigma &= \mathrm{ad}_J^{-1}\left\{\mathrm{i}\partial_x + \frac{\mathrm{i}}{3}\pi\left([q,J^\sigma]\right)\int_{\pm\infty}^x \mathrm{d}\,y \mathrm{tr}\,\left([q,.]J^{3-\sigma}\right) + \pi[q,.]\right\}.
\end{aligned}$$

Therefore

$$\Lambda e_\alpha = \lambda^3 e_\alpha, \qquad \Lambda = \Lambda_1\Lambda_2\Lambda_0.$$

- Expansion over the squared solutions and Fourier transform

Theorem 1 *The "squared" solutions fulfill the following completeness relations*

$$
\begin{aligned}
\delta(x-y)\Pi &= \frac{1}{2\pi}\sum_{a=1}^{6}(-1)^{a+1}\int_{l_a} \mathrm{d}\,\lambda\, \Big[e^{(a)}_{\beta_a}(x,\lambda)\otimes e^{(a)}_{-\beta_a}(y,\lambda) \\
&- e^{(a-1)}_{-\beta_a}(x,\lambda)\otimes e^{(a-1)}_{\beta_a}(y,\lambda)\Big] - \mathrm{i}\sum_{a=1}^{6}\sum_{n_a} \underset{\lambda=\lambda_{n_a}}{Res}\; G^{(a)}(x,y,\lambda).
\end{aligned}
$$

where

$$
\Pi = \sum_{\alpha\in\Delta^+}\frac{E_\alpha\otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha}{\alpha(J)}, \qquad G^{(a)}_{\beta_a}(x,y,\lambda) = e^{(a)}_{\beta_a}(x,\lambda)\otimes e^{(a)}_{-\beta_a}(y,\lambda).
$$

Hence any function X can be expanded over the "squared" solutions, namely

$$
X(x) = \frac{1}{2\pi}\sum_{a=1}^{6}(-1)^{a+1}\int_{l_a}\mathrm{d}\lambda\Big(X_{\beta_a}(\lambda)e^{(a)}_{-\beta_a}(x,\lambda) - X_{-\beta_a}(\lambda)e^{(a-1)}_{\beta_a}(x,\lambda)\Big)
$$

$$- \mathrm{i} \sum_{a=1}^{6} \sum_{n_a} X_{n_a},$$

the components of X are given by the expressions

$$X_{\beta_a}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\, y \langle \mathrm{ad}\,_J e^{(a)}_{\beta_a}(y,\lambda), X(y)\rangle$$

$$X_{-\beta_a}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\, y \langle \mathrm{ad}\,_J e^{(a-1)}_{-\beta_a}(y,\lambda), X(y)\rangle$$

$$X_{n_a} = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\, y \mathrm{tr}\,_1 \left(\mathrm{ad}\,_J \otimes \mathbb{1} \operatorname*{Res}_{\lambda=\lambda_{n_a}} G^{(a)}(x,y,\lambda) X \otimes \mathbb{1} \right).$$

In the case under consideration ($\mathfrak{g} = \mathfrak{sl}(3)$) simple poles of the resolvent are possible. Then the residues of $G^{(a)}(x,y,\lambda)$ are

$$\operatorname*{Res}_{\lambda=\lambda_{n_a}} G^{(a)}(x,y,\lambda) = \dot{e}^{(a)}_{\beta_a}(x,\lambda_{n_a}) \otimes e^{(a)}_{-\beta_a}(y,\lambda_{n_a}) + e^{(a)}_{\beta_a}(x,\lambda_{n_a}) \otimes \dot{e}^{(a)}_{-\beta_a}(y,\lambda_{n_a}).$$

where

$$e^{(a)}_{\alpha}(x,\lambda_a) = \lim_{\lambda\to\lambda_a} (\lambda - \lambda_a) e^{(a)}_{\alpha}(x,\lambda), \qquad \dot{e}^{(a)}_{\alpha}(x,\lambda_a) = \lim_{\lambda\to\lambda_a} \partial_\lambda (\lambda - \lambda_a) e^{(a)}_{\alpha}(x,\lambda).$$

Then the potential q admits the following expansion

$$q(x) = \frac{\mathrm{i}}{2\pi}\sum_{a=1}^{6}(-1)^{(a+1)}\beta_a(J)\int_{l_a}\mathrm{d}\,\lambda\left(s^{+}_{a,\beta_a}e^{(a)}_{\beta_a}(x,\lambda)+s^{-}_{a,-\beta_a}e^{(a-1)}_{-\beta_a}(x,\lambda)\right)$$
$$-\mathrm{i}\sum_{a=1}^{6}\sum_{\alpha\in\Delta_a^+}\left(\dot{q}^{(a)}_{\alpha}(\lambda_a)e^{(a)}_{\alpha}(x,\lambda_a)+q^{(a)}_{\alpha}(\lambda_a)\dot{e}^{(a)}_{\alpha}(x,\lambda_a)\right),$$

where

$$q^{(a)}_{\alpha}(\lambda_a) = \int_{-\infty}^{\infty}\mathrm{d}\,y\langle \mathrm{ad}\,_J q(y), e^{(a)}_{-\alpha}(y,\lambda_a)\rangle,$$
$$\dot{q}^{(a)}_{\alpha}(\lambda_a) = \int_{-\infty}^{\infty}\mathrm{d}\,y\langle \mathrm{ad}\,_J q(y), \dot{e}^{(a)}_{-\alpha}(y,\lambda_a)\rangle.$$

It is derived from the Wronskian relation

$$(\hat{\chi}^a J\chi^a - J)|_{-\infty}^{\infty} = \mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\,x\hat{\chi}^a[J,q]\chi^a.$$

One can easily check that

$$\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}\,x\langle\hat{\chi}^a[J,q]\chi^a, E_{-\alpha}\rangle = -\mathrm{i}\,[\![q, e_{-\alpha}^{(a)}]\!],$$

where

$$[\![X,Y]\!] \equiv \int_{-\infty}^{\infty}\mathrm{d}\,x\langle X, [J,Y]\rangle$$

is the so-called skew-skalar product.

On the other hand, we have

$$\langle(\hat{\chi}^a J\chi^a - J)|_{-\infty}^{\infty}, E_{-\alpha}\rangle = -\alpha(J)s_{a,\alpha}^{+}.$$

By analogy, the variation of q can be expanded in the following manner

$$\mathrm{ad}\,_J^{-1}\delta q(x) = \frac{\mathrm{i}}{2\pi}\sum_{a=1}^{6}(-1)^a\int_{l_a}\mathrm{d}\,\lambda\left(\delta s_{a,\beta_a}^{+}e_{\beta_a}^{(a)}(x,\lambda) - \delta s_{a,-\beta_a}^{-}e_{-\beta_a}^{(a-1)}(x,\lambda)\right).$$

The latter is obtained starting from another Wronskian relation

$$\hat{\chi}^a \delta \chi^a |_{-\infty}^{\infty} = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}\, x \hat{\chi}^a \delta q \chi^a.$$

- Description of NEE of Kaup-Kuperscmidt type via recursion operators

  It can be verified that the integrable hierarchy of Kaup-Kuperscmidt equation in terms of Λ operator reads

$$\mathrm{iad}\,_J^{-1} \partial_t q = \sum_{l=1}^{n} c_{3l-1} \Lambda^{l-1} \Lambda_0 \Lambda_1 \mathrm{ad}\,_J^{-1} [q, J^2] - \sum_{l=1}^{n} c_{3l-2} \Lambda^{l-1} \Lambda_0 q, \;\; N = 3n-1,$$

$$\mathrm{iad}\,_J^{-1} \partial_t q = \sum_{l=1}^{n-1} c_{3l-1} \Lambda^{l-1} \Lambda_0 \Lambda_1 \mathrm{ad}\,_J^{-1} [q, J^2] - \sum_{l=1}^{n} c_{3l-2} \Lambda^{l-1} \Lambda_0 q, \;\; N = 3n-2,$$

  where

$$f(\lambda) = \sum_{m=1}^{N} c_m \lambda^m, \qquad c_{3l} = 0, \quad l = 0, 1, 2, \ldots.$$

In particular, for the Kaup-Kuperschmidt equation itself we have $f(\lambda) = \lambda^5$ and therefore

$$\mathrm{i}\,\mathrm{ad}_J^{-1}\partial_t q - \Lambda\Lambda_0\Lambda_1\mathrm{ad}_J^{-1}[q, J^2] = 0.$$

After substituting the expansions of q and its variation one obtains

$$\mathrm{i}\partial_t s^{\pm}_{a,\beta_a} \mp \lambda^5\beta_a(J^2)s^{\pm}_{a,\beta_a} = 0 \qquad \Rightarrow \qquad s^{\pm}_{a,\beta_a} = s^{\pm}_{a,\beta_a,0}\exp\left(\mp\mathrm{i}\beta_a(J^2)\lambda^5 t\right).$$