ALGEBRAIC METHODS OF LATTICE MANY-BODY SYSTEMS

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This talk is a short review on systems

with self-organized dynamics

The original results are joint work with J. Brankov

arXiv:1101.2822, to appear in TMPh

Outline

1. SELF-ORGANIZED CRITICALITY AND STOCHASTIC DYNAMICS
2. SAND PILE MODELS
3. DIRECTED ABELIAN ALGEBRAS AND APPLICATIONS
4. STATIONARY STATE AND AVALANCHE EVOLUTION
5. EXTENDING THE RESULTS TO 2 DIMENSIONS

SELF-ORGANIZED CRITICALITY

AVALANCHE CASCADE PROCESSES

WIDE RANGE of APPLICATIONS IN DIVERSE AREAS -

Planetary Dynamics, Life Dynamics, Stellar Dynamics

GAS DISCHARGE, FOREST FIRES, LAND/SNOW SLIDING,

EXTINCTION of SPECIES in BIOLOGY, BRAIN ACTIVITY

EARTHQUAKES, VOLCANOES, STAR FORMATION,

METEORITE SIZE DISTRIBUTION, RIVER NETWORKS

PROCESSES in FINANCE and STOCK MARKET

SELF-ORGANIZED CRITICALITY is due to

LONG-RANGED SPACE-TIME CORRELATIONS in

NONEQUILIBRIUM STEADY STATES of

SLOWLY DRIVEN SYSTEMS without FINE TUNING of

ANY CONTROL PARAMETER

An external agent SLOWLY drives the system and

through successive relaxation events a burst of

of activity - cascade process, avalanche -

starts within the system itself.

The SYSTEM becomes CRITICAL under its own

DYNAMICAL EVOLUTION due to EXTERNAL AGENT

SLOW DRIVE of THE SYSTEM by ENERGY, MASS INPUT

(MAY ALSO BE the SLOPE, LOCAL VOIDS)

LIMITED ENERGY STORAGE CAPACITY of MANY-BODY SYSTEM

MASS BECOMES LOCALLY TOO LARGE (LOCALLY OVERHEATED)

and is REDISTRIBUTED - TRANSPORT PROCESS STARTS

SELF ORGANIZING DYNAMICS

GOVERNED by POWER LAWS

TWO TIME SCALES WIDE SEPARATED

DRIVE TIME SCALE - MUCH SLOWER RATE

RELAXATION TIME SCALE - SHORT TIME

THRESHOLD - above it CASCADE of TOPPLINGS PROPAGATES

SURPLUS of MASS, ENERGY is DISSIPATED

through SYSTEM'S BOUNDARY

The SAND PILE model - PARADIGM

for SELF ORGANIZING DYNAMICS

analogously to

the OSCILLATOR in QUANTUM MECHANICS

the ISING MODEL in STATISTICAL PHYSICS

the ASEP - the FUNDAMENTAL MODEL of

NONEQUILIBRIUM PHYSICS

The concept SELF-ORGANIZED CRITICALITY SOC

introduced by Bak, Tang and Wiesenfeld (1987)

ABELIAN SANDPILE MODEL ASPM

to illustrate their idea of complexity of a system of many elements

Sand pile is formed on a horizontal circular base with any arbitrary initial distribution of sand grains. Steady state - sand pile of conical shape, formed by slowly adding (external drive) sand grains, one after another. Constant angle of the surface with the horizontal plane. Addition of grains drives the system to a critical point - sand avalanche propagates on the pile surface.

BTW ASPM - defined on d dimensional lattice (on any graph)

site i of the lattice is occupied by a number of sand grains

associated characteristics - height h_i; critical value h_{crit}

$h_i < h_{crit}$ stable site
$h_i \geq h_{crit}$ unstable site

UNSTABLE SITE TOPPLES - dissipates energy

REDISTRIBUTES GRAINS TO THE NEIGHBOUR SITES

DIFFERENT SAND PILE MODELS DIFFER in the TOPPLING RULES

DETERMINISTIC SPM - the number of grains transmitted from a site i to j are fixed, (BTW -1987, Dhar -1999)

STOCHASTIC SPM - sites where grains are redistributed are chosen at random, (Manna - 1991, Paczuski, Bassler-2000, Kloster, Maslov, Tang - 2001)

ABELIAN PROPERTY - FINAL STABLE CONFIGURATION is

INDEPENDENT of the ORDER of ADDING the GRAINS

If in a stable configuration C a particle is added first at a site i, then at a site j - the final stable configuration is the same, if a particle is first added at a site j, then at a site i

DIRECTED ABELIAN MODELS - redistribution in a fixed direction(s)

Application of DIRECTED ABELIAN ALGEBRAS

correspond to DIRECTED GRAPHS

with each site of the L dimensional lattice a generator a_i
of an Abelian algebra is associated

Alcaraz, Rittenberg, Phys.Rev.E78 (2008)

MAIN CHARACTERISTICS

- SIZE s - total number of topplings

- AREA a - total number of sites that topple

- LIFE TIME t - duration, length,
short virtual time

- WIDTH x - radius or maximum distance
of a toppled site from the origin

these quantities are not independent

related to each other by scaling laws

FINITE-SIZE SCALING

SCALE INVARIANCE - POWER LAWS ARE DIRECT CONSEQUENCE

lower bound - size of smallest element (one grain)

upper bound - through dissipation at the border
size, area, duration are limited

CUT OFF at the UPPER BOUND described by the

SCALING HYPOTHESES (LAWS)

$$
\begin{aligned}
P(s) &= s^{-\sigma_s} f(s_c) \\
P(t) &= s^{-\sigma_\tau} g(t_c) \\
P(x) &= s^{-\sigma_x} h(x_c)
\end{aligned}
$$

$\sigma_s \quad \sigma_\tau \quad \sigma_x$ CRITICAL EXPONENTS
define the UNIVERSALITY CLASS

s_c, t_c, x_c CUT OFF PARAMETERS

in the limit $L \to \infty$ $s_c \sim L^D, t_c \sim L^z, x_c \sim L^{1/\zeta}$

D - FRACTAL DIMENSION of the AVALANCHE CLUSTER

z, ζ - DYNAMICAL EXPONENTS

THE EXPONENTS - NOT INDEPENDENT

PROBABILITY CONSERVATION - for any two AVALANCHE CHARACTERISTICS (y_1, y_2) and corresponding dynamical exponents one has

$$\frac{\sigma_{y_1} - 1}{\sigma_{y_2} - 1} = \frac{D_{y_2}}{D_{y_1}}$$

$D_y(\sigma_y - 1)$ IS AN INVARIANT

DSPM - $z = 1$

$$\sigma_\tau - 1 = D(\sigma_s - 1) = (\sigma_x - 1)/\zeta \qquad D = \sigma_\tau$$

Numerical and analytical results for critical exponents

DETERMINISTIC - $\sigma_s = 1.43, \quad D = \sigma_\tau = 3/2$

STOCHASTIC - $\sigma_s = 1.43, \quad D = \sigma_\tau = 7/4$

RECENT - Alcaraz and Rittenberg

$D = \sigma_\tau = 1.78 \pm 0.01$

DAA FORMALISM on L-site 1 DIMENSIONAL LATTICE

generators a_i, $i = 1, 2, ..., L$

$$[a_i, a_j] = 0$$

QUADRATIC ALGEBRA

$$a_i^2 = \mu a_{i+1}^2 + (1-\mu) a_i a_{i+1}$$

BC $\quad a_L^2 = \mu + (1-\mu) a_L \quad (a_{L+1} = 1)$

The algebra is semisimple - all representations are decomposable into irreducible representations.

The irreducible representations are one dimensional.

The regular representation has dimension 2^L and this is the number of irreducible representations.

Basis of the regular representation - the 2^L monomials

$$1, a_i, a_i a_j, ..., a_1 a_2 ... a_{L-1} a_L$$

Map the regular representation vector space on L-site chain

one particle at a site i- if a_i appears in the monomial
empty site i - otherwise
hence - 2^L configurations

a_i act on the regular representation and can be diagonalized simultaneously; common eigenvalue 1
a_L has eigenvalues $1, \mu$

$$a_i \Phi = \Phi, \quad i = 1, 2, ..., L$$

STATIONARY STATE Φ

$$\Phi = \prod_{i=1}^{L} \frac{\mu + a_i}{1+\mu}$$

a site is occupied with probability $\frac{1}{1+\mu}$
a site is empty with probability $\frac{\mu}{1+\mu}$

Physical meaning of the quadratic relation

$$a_i^2 = \mu a_{i+1}^2 + (1-\mu) a_i a_{i+1}$$

$h_c = 2$, if $h_c(i) \geq 2$ - with a probability μ
two particles move to site $i+1$
and with probability $1-\mu$
one particle moves to $i+1$, one stays at i

AVALANCHE EVOLUTION

- adding 2 grains at the first site
defined by the ACTION of a_1^2 on the steady state

$$a_1^2 \prod_{i=2}^{L} \frac{\mu + a_i}{1+\mu} = (\mu + (1-\mu)a_1) \prod_{i=2}^{L} \frac{\mu + a_i}{1+\mu}$$

subsequent action

$$RHS = \left[\frac{\mu(1-\mu)a_1 a_2}{1+\mu} + \frac{\mu a_2^3 + \mu^2 a_2^2 + (1-\mu)a_1 a_2^2}{1+\mu} \right] \prod_{i=3}^{L} \frac{\mu + a_i}{1+\mu}$$

and with $a_i = 1$ for all a_i left behind the avalanche front

$$a_i^n \frac{\mu + a_i}{1+\mu} \hat{=} \frac{1}{(1+\mu)^2} \left[\mu a_{i+1}^{n+1} + (1+\mu^2) a_{i+1}^n + \mu a_{i+1}^{n-1} \right]$$

The virtual time evolution $\tau \geq 2$

$$a_1^2 \prod_{i=2}^{L} \frac{\mu + a_i}{1+\mu} \hat{=} \sum_{n=1}^{\tau} P_n(\tau) a_\tau^n \prod_{k=\tau+1}^{L} \frac{\mu + a_k}{1+\mu}$$

$P_n(\tau)$ - PROBABILITY for the AVALANCHE to take place at VIRTUAL TIME τ with n GRAINS at SITE $i = \tau$

Recurrent relations for $P_n(\tau)$

$$\begin{aligned}
P_1(\tau) &= R_-^{(2)} P_2(\tau - 1), \\
P_2(\tau) &= R_0^{(2)} P_2(\tau - 1) + R_-^{(3)} P_3(\tau - 1), \\
P_n(\tau) &= R_-^{(n+1)} P_{n+1}(\tau - 1) + R_0^{(n)} P_n(\tau - 1) \\
&+ R_+^{(n-1)} P_{n-1}(\tau - 1)
\end{aligned}$$

$2 \leq n \leq \tau, \qquad P_n(1) = \delta_{n,2}$

$$R_+^{(n)} = R_-^{(n)} = \frac{\mu}{(1+\mu)^2}, \quad R_0^{(n)} = \frac{1+\mu^2}{(1+\mu)^2}$$

$$R_+^{(n)} + R_0^{(n)} + R_-^{(n)} = 1$$

RANDOM WALKER at time τ stays
at position n with probability $\frac{1+\mu^2}{(1+\mu)^2}$
moves to positions $n+1$ or $n-1$ with probability $\frac{\mu}{(1+\mu)^2}$

Probability for duration τ avalanche is the

FIRST PASSAGE PROBABILITY

at virtual time τ to return to initial position $n = 1$
(discrete coordinates: virtual time τ, space n)

form an

$$p(T) = P_1(T) \sim \frac{1}{\sqrt{DT^3}} \approx \frac{1}{T^{\sigma_\tau}}$$

CRITICAL EXPONENT $\sigma_\tau = 3/2$

RANDOM WALKER UNIVERSALITY CLASS

IN ONE DIMENSION

DETERMINISTIC and STOCHASTIC AVALANCHE

BELONG to the SAME UNIVERSALITY CLASS

TWO DIMENSIONS

rotated by $\pi/4$ square lattice $i,j,\ i,j=1,2,...,L$
DAA of Alcaraz and Rittenberg

$$\begin{aligned} a_{i,j}^2 &= \alpha\left(\mu a_{i+1,j}^2 + (1-\mu)a_{i,j}a_{i+1,j}\right) \\ &+ (1-\alpha)\left(\mu a_{i,j+1}^2 + (1-\mu)a_{i,j}a_{i,j+1}\right) \end{aligned}$$

Monte Carlo simulations - critical exponent

$$\sigma_\tau = 1.78 \pm 0.01$$

CONTRADICTION to PREVIOUSLY determined VALUE

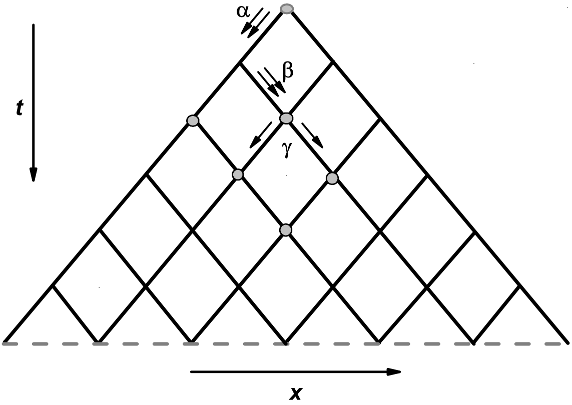
$$\sigma_\tau = 1.75$$

CONSIDER DAA on a rotated square lattice
sites form the triangular array $\mathcal{L} = (i, j), i = 1, ..., T; j = 1, ..., i$,
i labels the integer step τ, j numbers the sites visited
at time $\tau = i$ in the horizontal (spacial) direction;
T is the avalanche size in temporal direction

$$a^2_{i,j} = \alpha a^2_{i+1,j} + \beta a^2_{i+1,j+1} + \gamma a_{i+1,j} a_{i+1,j=1}$$

$$\alpha + \beta + \gamma = 1$$

$h_i \geq 2$ unstable site - relaxes by multiple (successive)
2-particle topplings to the left (right) neighbour in front
with probability α (β) and
one - left, one -right with probability γ



Фигура: Schematic representation of the rotated by $\pi/4$ square lattice and the directed toppling rules. The bottom boundary of the lattice is open.

A SITE CAN EMIT ONLY EVEN NUMBER OF GRAINS,
BUT RECEIVES ANY NUMBER

On the open boundary generators satisfy

$$a_{T,j}^2 = 1, \qquad j = 1, 2, ..., T$$

stationary state

$$\Phi_{1,T} = \prod_{i=1}^{T} \prod_{j=1}^{i} \frac{1 + a_{i,j}}{2}$$

$$a_{i,j}\Phi_{1,T} = \Phi_{1,T}, \quad (i,j) \in \mathcal{L}$$

The avalanche evolution starts by

$$
\begin{aligned}
a_{1,1}^2 \Phi_{2,T} &= \\
(\alpha a_{2,1}^2 &+ \beta a_{2,2}^2 + \gamma a_{2,1} a_{2,2}) \frac{1 + a_{2,1}}{2} \frac{1 + a_{2,2}}{2} \Phi_{3,T}
\end{aligned}
$$

to describe evolution one needs NUMBER of PARTICLES

TRANSFERRED from TIME STEP τ to $\tau + 1$.
a layer τ emits even number - hence

$$
a_{i,j}^{2p} \frac{1 + a_{i,j}}{2} \hat{=} \sum_{k=0}^{2p} C_k^{(2p)} a_{i+1,j}^{2p-k} a_{i+1,j+1}^k ,
$$

AVALANCHE EVOLUTION

$$a^2_{1,1}\Phi_{2,T} \hat{=} \sum_{n=0}^{n_{max}(\tau)} \left[\sum_{n_1+...+n_\tau=n} P(n_1,...,n_\tau|\tau) \prod_{k=1}^{\tau} a^{n_k}_{\tau,k} \right] \Phi_{\tau+1,T}$$

$$\tau = 2,...,T-1$$

$P(n_1,...,n_\tau|\tau)$ - PROBABILITY that at time $i=\tau$

the SITES $(\tau,1),(\tau,2),...,(\tau,\tau)$ have

OCCUPATION NUMBERS $n_1,n_2,...,n_\tau$

$$n_1+n_2+...+n_\tau = 0,1,...,n_{max}(\tau)$$

MONOMIAL

$\prod_{k=1}^{\tau} a_{\tau,k}^{n_k}$ shows

DISTRIBUTION of PARTICLES at row τ

FLUX OF PARTICLE to next row $\tau+1$ is obtained

by applying the action of $a_{\tau,i}^{2p_i}$ whose form

two types of terms - passive component and active component

$$a_{i,j}^{2p} \hat{=} \Sigma_{2p}^{even} + \Sigma_{2p-2}^{even} a_{i+1,j} a_{i+1,j+1}$$

Recurrent relations for the probabilities

$P(n_1, ..., n_\tau | \tau)$ - OPEN PROBLEM

written - only up to $\tau = 3$

IMPORTANT CHARACTERISTICS

MAXIMUM CURRENT $I_{max}(\tau)$

MAXIMUM HIGHT $h_{max}(\tau)$

MAXIMUM CURRENT

of particles leaving a row τ

$$
\begin{aligned}
I_{max}(\tau) &= \frac{\tau^2+1}{2}+1, \qquad \tau \quad odd \\
I_{max}(\tau) &= \frac{\tau^2}{2}+2, \qquad \tau \quad even
\end{aligned}
$$

result is based on

- configurations with all sites occupied

- recurrent relations

$$
I_{max}(\tau) - I_{max}(\tau-2) = 2\tau - 2, \quad \tau > 2
$$

Global maximum of hight is reached

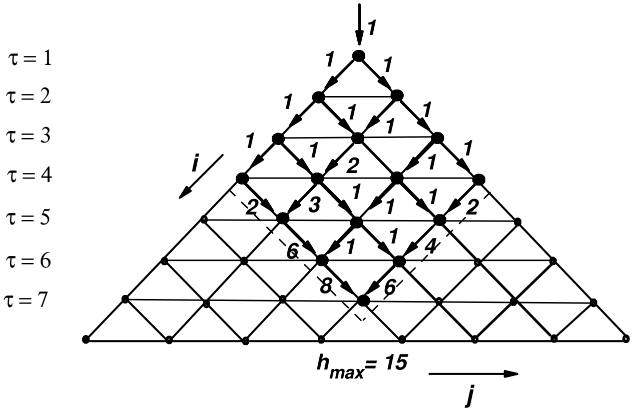
- odd $\tau = 2n - 1$ at central site $(2n - 1, n)$

$$h_{max}(\tau, (\tau + 1)/2) = (\tau^2 - 1)/4 + 3$$

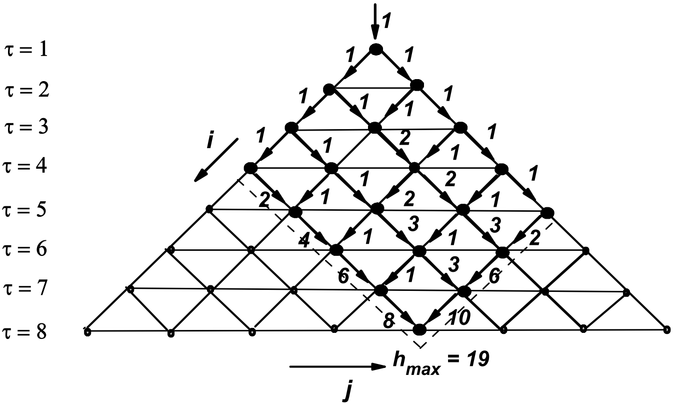
- even $\tau = 2n$ at central site $(2n, n)$, $(2n, n + 1)$

$$h_{max}(\tau) = \frac{\tau^2}{4} + 3, \qquad \tau/2 \quad even$$

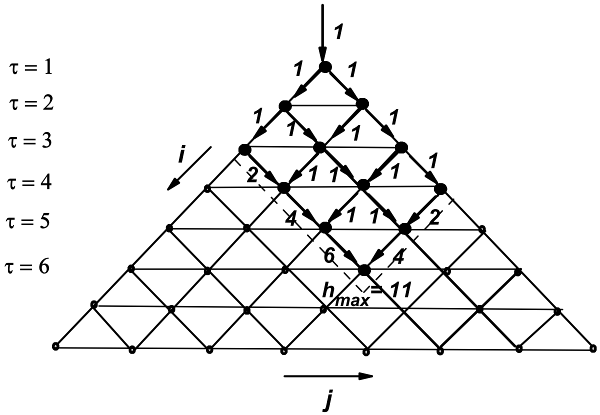
$$h_{max}(\tau) = \frac{\tau^2}{4} + 2, \qquad \tau/2 \quad odd$$



Фигура: Schematic illustration of an avalanche leading to a maximum unstable at the central site of an odd-τ row. The integers besides the arrows indicate the number of particles transferred in the corresponding direction



Фигура: Schematic illustration of an avalanche leading to a maximum unstable
at a central site of an even-τ row when $\tau/2$ is even. The integers besides the arr
indicate the number of particles transferred in the corresponding direction



Фигура: The same as in Fig. 3 for even-τ row when $\tau/2$ is odd.

ATTEMPT to extend DAA (Alc, Rit,) to 2-dim. STOCHASTIC DSM

the considered QUADRATIC ALGEBRA CORRESPONDS to

ANALYTICALLY STUDIED STOCHASTIC TOPPLING RULES
(M.Paczuski, K.Bassler; M.Kloster, S.Maslov, C.Tang)

and predicted a consistent set of critical exponents

Within DAA we suggest virtual time evolution of 2-dim

DIRECTED STOCHASTIC AVALANCHES from which the probability

distribution of avalanche duration can be derived.