Topological Quantization of Free Massive Bosonic Field

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ʼΑγεωμέτρητος μηδὲὶς εἰσίτω

Schematic view of topological quantization

1 Take a physical system to analyze.

2 Construct an equivalent geometrical configuration.

3 Calculate the topological invariants.

4 Analyze what this imply in the physical system .

The physical system: Bosonic Field

Axioms in QM

- The state of the system is represented by a vector in Hilbert space
- Observables are represented by hermitian operators
- The mesaurement of an observable yields one of its eigenvalues as a result
- The time evolution of the state of the system is governed by the Shrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle$$

Klein-Gordon equation

(Shrödinger equation + Special Relativity)

$$S = \int d^4x \, \mathcal{L}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \tfrac{1}{2} m^2 \varphi^2$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(+\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

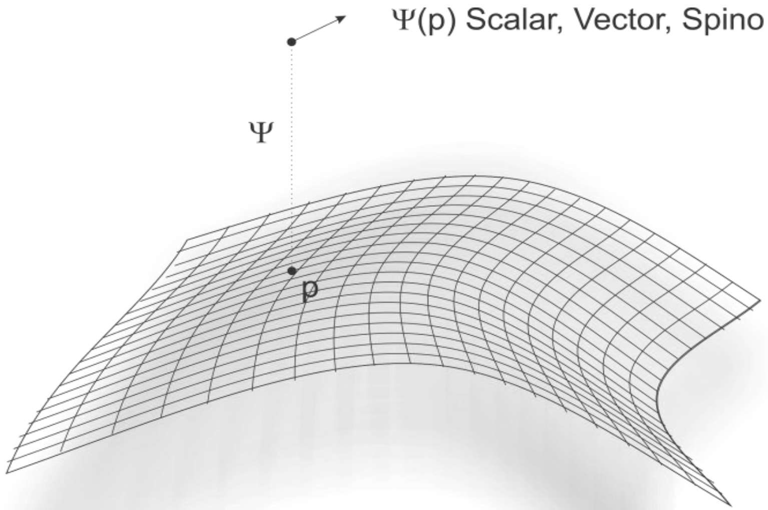
$$0 = \delta S \longrightarrow (-\partial^2 + m^2)\varphi = 0$$

$$\partial^2 \equiv \partial^\mu \partial_\mu = -\partial_0^2 + \nabla^2$$

$$\Rightarrow \varphi(\mathbf{x}, t) = \int \frac{d^3k}{f(k)} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} + a^*(-\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} + i\omega t} \right]$$

Bose-Einstein fields

Quantum Field: Set of operators $\varphi(\mathbf{x})$ defined at each point $\mathbf{x}$ in space ($\varphi(\mathbf{x},t) = e^{iHt/\hbar}\varphi(\mathbf{x},0)e^{-iHt/\hbar}$ or space-time)



$$\varphi(\mathbf{x},t) = \int \frac{d^3k}{f(k)}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} + a^*(-\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right]$$

$$[\varphi(x),\varphi(y)] = [\pi(x),\pi(y)] = 0$$

$$[\varphi(\xi),\pi(y)] = i\delta^{(3)}(x-y)$$

$$H = \int \mathcal{H}\, d^3x$$

$$P^i = -\int \pi\,\partial_i\varphi\, d^3x$$

$|0\rangle$ Vacuum state (state of "no particles")

$a^+(p_1)|0>$ State of 1 particle with moment p_1

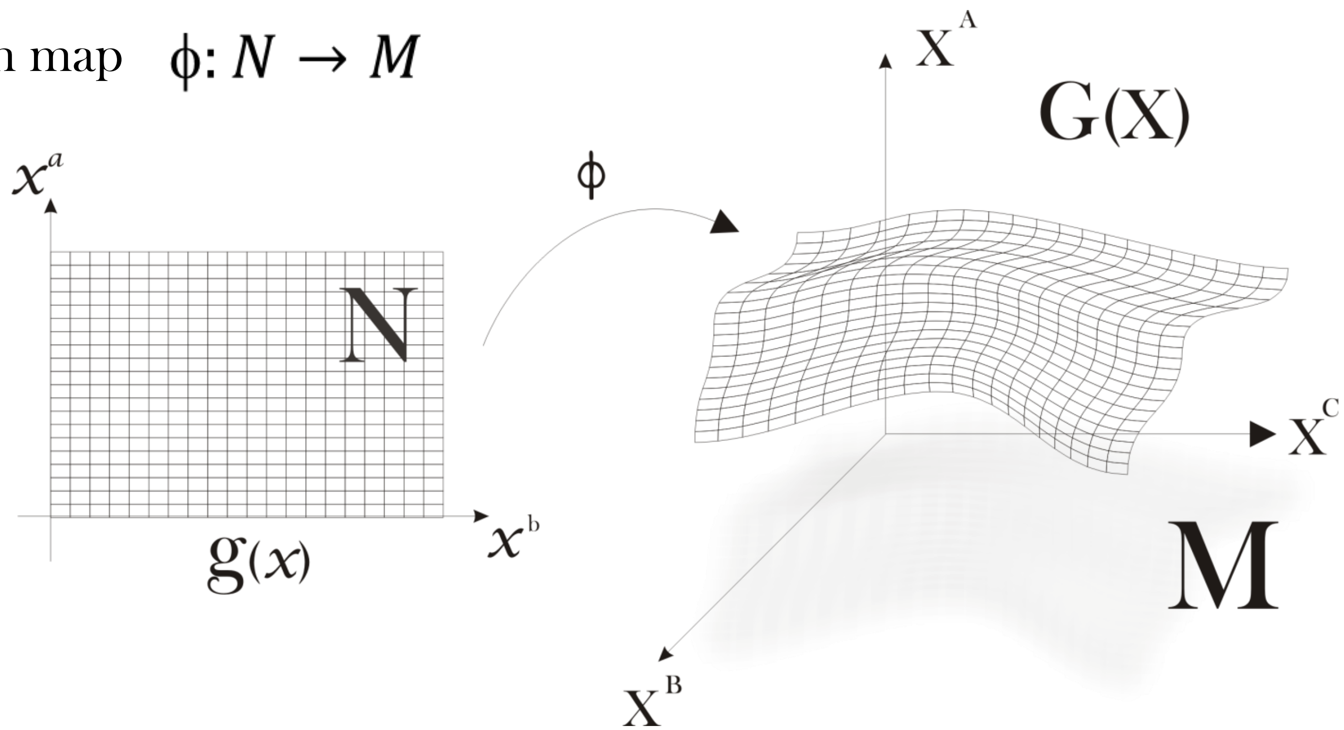
$a^+(p_1)a^+(p_1)|0>$ State of 1 particle with moment p_1 and other particle with moment p_2

1 Physical system
2 Geometrical configuration
3 Topological invariants
4 Analyze spectrum

Harmonic mapping

Manifolds	(N, g)	(M, G)
Metrics	$g = g(x)$	$G = G(X)$
Coordenates	$x^a \ (a = 1, \dots, n)$	$X^A \ (A = 1, \dots, m)$

Smooth map $\phi: N \to M$



Action of harmonic maps

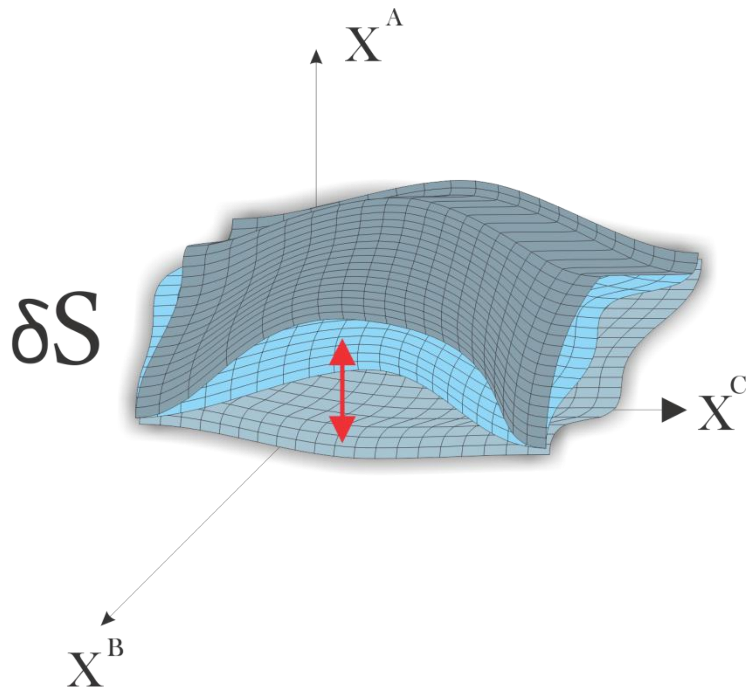
$$S = \int d^n x \sqrt{|\det(g)|}\, g^{ab}(x) \frac{\partial X^A}{\partial x^a} \frac{\partial X^B}{\partial x^b} G_{AB}$$

If the metric g_{ab} is the induced metric:

$$g_{ab} = \frac{\partial X^A}{\partial x^a} \frac{\partial X^B}{\partial x^b} G_{AB}$$

⇨ $$S = \int d^n x \sqrt{|\det(g)|}$$

⇨ δS=0 Minimun embedding!



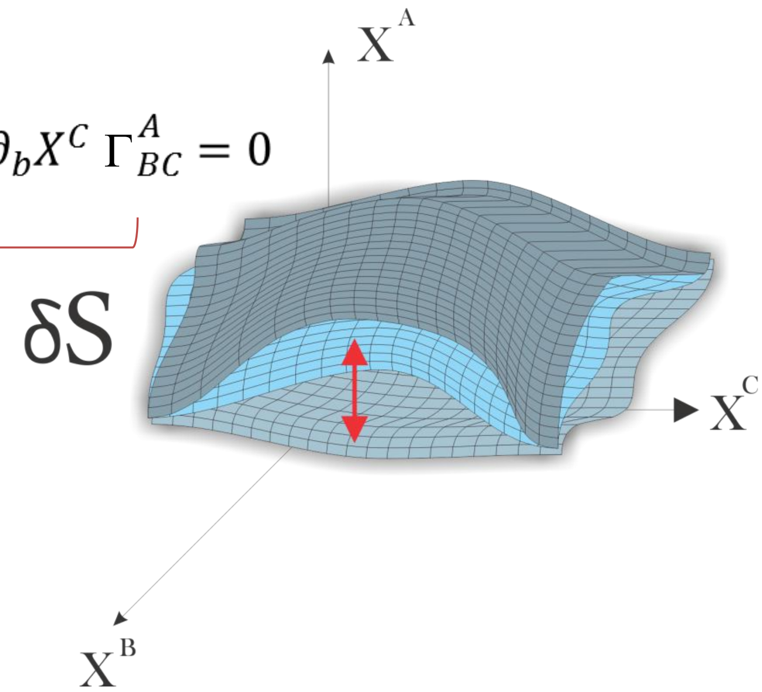
Field as an harmonic map

$$S = \int d^n x \sqrt{|\det(g)|}\, g^{ab}(x) \frac{\partial X^A}{\partial x^a} \frac{\partial X^B}{\partial x^b} G_{AB}$$

$\delta S=0$ ➡ $\partial_b\left(\sqrt{|\det(g)|}\, g^{ab} \partial_a X^A\right) - \sqrt{|\det(g)|}\, g^{ab} \partial_a X^B \partial_b X^C\, \Gamma^A_{BC} = 0$

$\sqrt{|\det(g)|}\, \nabla^2 X^A$

$m^2 X^A$



Klein-Gordon Eq.: $(-\partial^2 + m^2)\varphi = 0$

$$\Gamma^i_{jk} = \frac{1}{2} G^{is}(\partial_j G_{sk} + \partial_k G_{js} - \partial_s G_{jk})$$

¿ $g = g(x)$, $G = G(X)$ → $S = \int d^n x \left(-\frac{1}{2}\partial^a X^A \partial_a X_A - \frac{1}{2} m^2 X^A X_A\right)$?

Metric of a free massive bosonic field

$$ds^2 = -2\,dX^+dX^- - \mu^2\left(\sum_{I=1}^{8} X^I X_I\right)(dX^+)^2 + \sum_{I=1}^{8} dX^I dX^I$$

↓

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, g^{ab}\, G_{AB}\,\partial_a X^A \partial_b X^B \qquad \sigma^a = \tau, \sigma$$

↓

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, g^{ab}(-\partial_a X^+\partial_b X^- - \partial_a X^-\partial_b X^+ - \mu^2 X_I^2\,\partial_a X^+\partial_b X^+ + \partial_a X^I\partial_b X^I)$$

$$\sqrt{-g}g^{ab} = \eta^{ab}, \qquad -\eta_{\tau\tau} = \eta_{\sigma\sigma} = 1 \quad + \quad X^+ = \alpha' p^+ \tau, \quad p^+ > 0$$

→

$$S_{l.c.}^{bos.} = \frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma\,[\partial_\tau X^I\partial_\tau X^I - \mu^2 X_I^2 - \partial_\sigma X^I\partial_\sigma X^I]$$

Inspiration:
Dirac’s electric charge quantization

$$\vec{B} = \frac{g}{4\pi r^2}\vec{r}$$

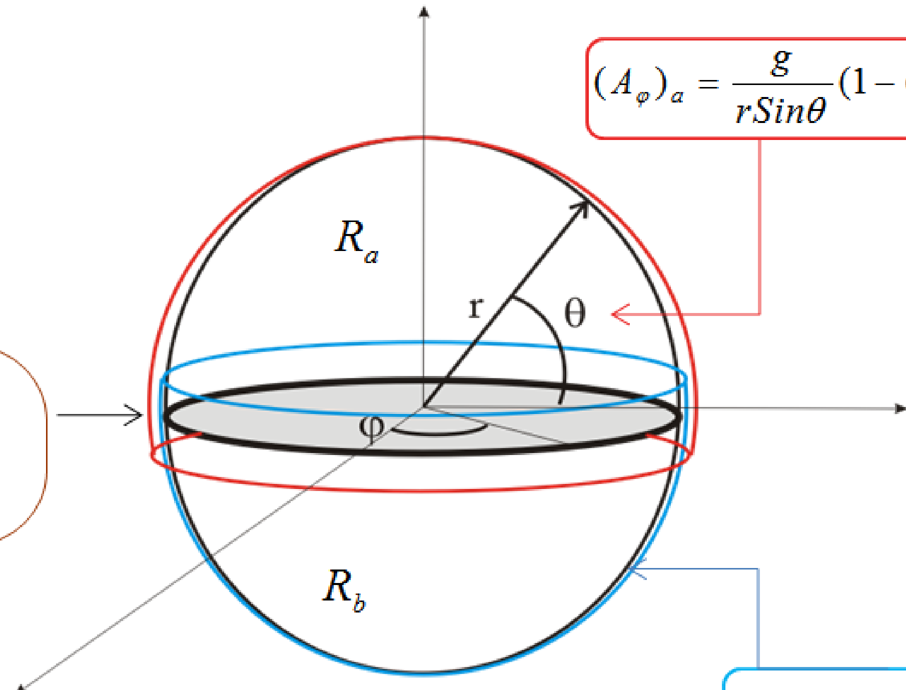
$$A_t = A_r = A_\theta = 0$$

$$(A_\mu)_b = (A_\mu)_a + \frac{\hbar c}{e}\partial_\mu \alpha$$

$$\Psi' = e^{-i\alpha}\Psi$$

$$e^{-i\alpha} = \exp\left(\frac{2ige}{\hbar c}\varphi\right)$$

$$(A_\varphi)_a = \frac{g}{r \mathrm{Sin}\theta}(1 - \mathrm{Cos}\theta)$$



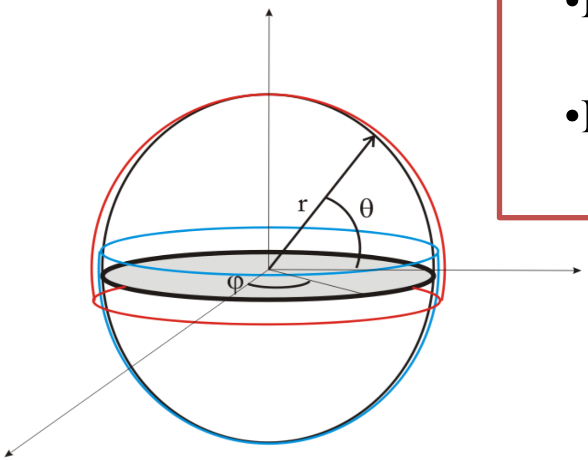
$$(A_\varphi)_b = -\frac{g}{r \mathrm{Sin}\theta}(1 + \mathrm{Cos}\theta)$$

$$\frac{2ge}{\hbar c} = n,\ n \in \mathbb{Z}$$

Topological quantization

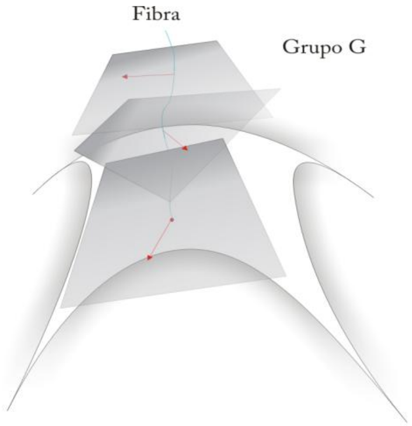
- Characteristic Class of a principal bundle P over M: $\int C(P) = \xi$

- Topological spectrum: $f(a_1, a_2, \ldots, a_c) = \xi, \qquad \xi \in \mathbb{Z}$



- Induced by the conexión (Dirac)

- Intrinsic (Tangent bundle)



3 Topological invariants

1 Physical system

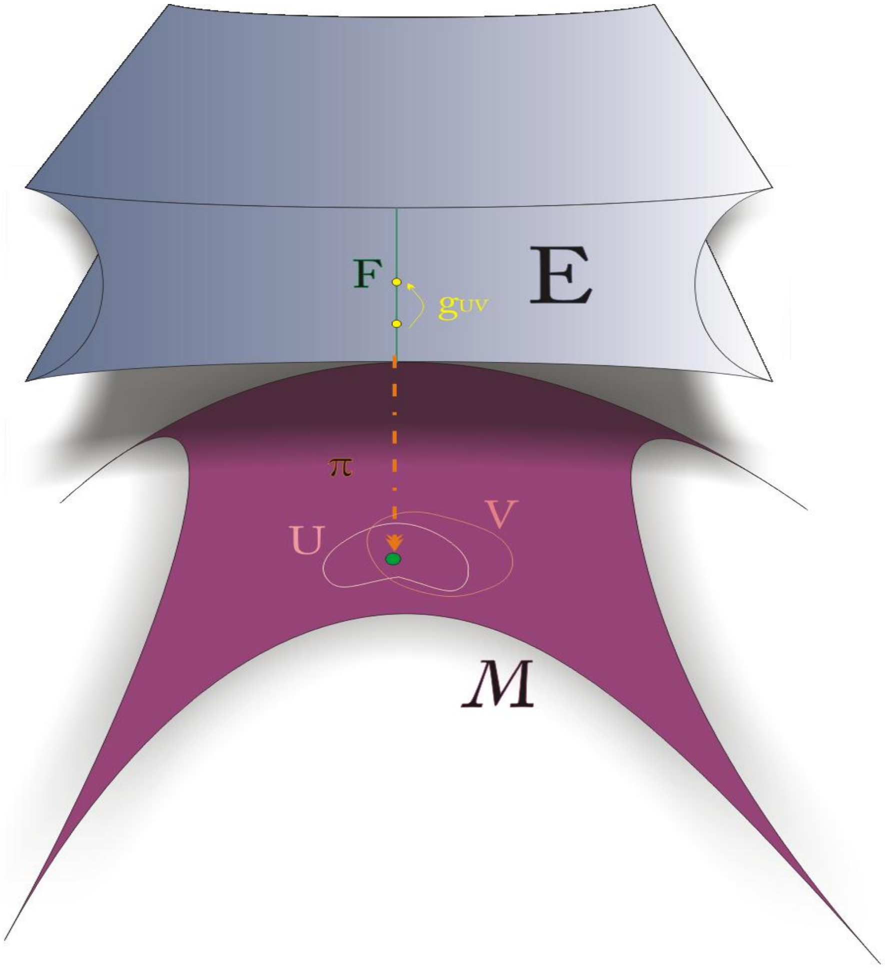
2 Geometrical configuration

4 Analyze spectrum

Fiber bundle

- A fiber bundle is defined by:
  - Total space E
  - Base space M
  - Fiber F
  - Projection π
  - Group G

$$(E, \pi, M, F, G)$$



Construction of a fiber bundle

•Riemannian manifold (M, g)

•Atlas (U_α, ϕ_α)

$$\cup_\alpha U_\alpha = M$$

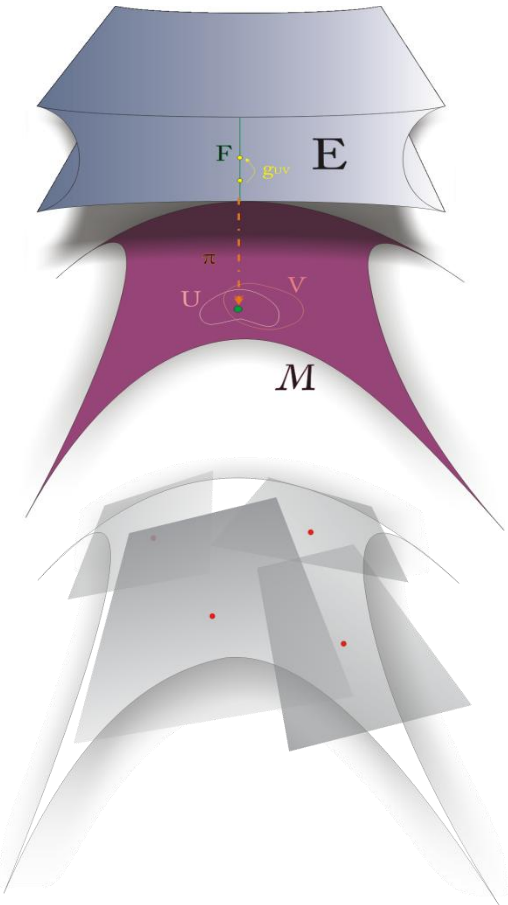
•Lie group G with Lie algebra $\mathfrak{g}$ → $h_{ij} : U_i \cap U_j \mapsto G$

➡ $\omega_i = h_{ji}^{-1} \omega_j h_{ij} + h_{ji}^{-1} dh_{ij}$

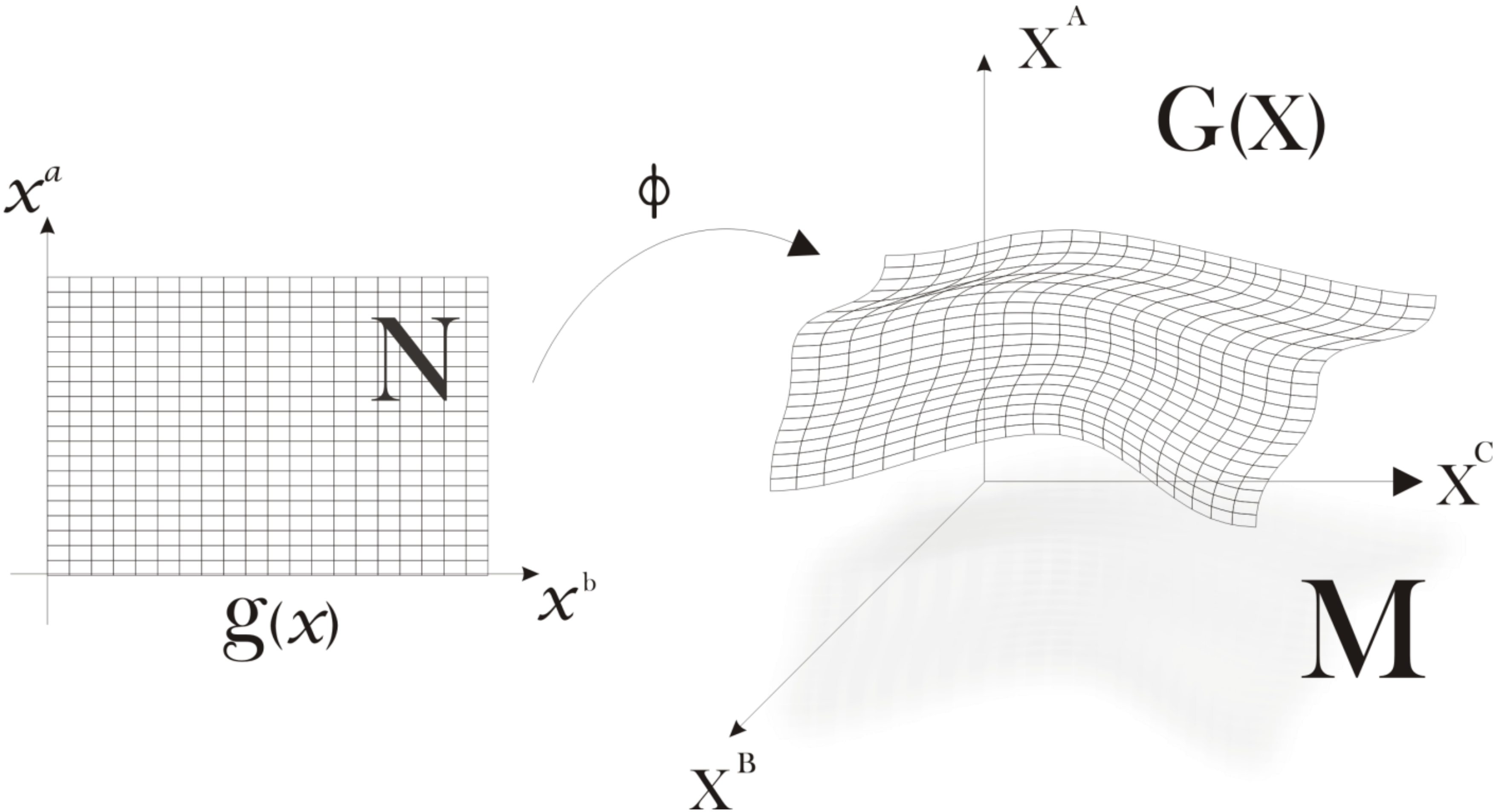
Principal Fiber Bundle P

dim P = dim M + dim G

$\int C(P) = \xi$ ➡ $f(a_1, a_2, \ldots, a_c) = \xi, \qquad \xi \in \mathbb{Z}$



Two fiber bundles



Topological quantization for G_{AB}

$$ds^2 = -2 dX^+ dX^- - \mu^2 \left(\sum_{I=1}^{8} X^I X_I \right) (dX^+)^2 + \sum_{I=1}^{8} dX^I dX^I$$

$G_{-+} = G_{+-}$

G_{++}

$G_{IJ} = \delta_{IJ}$

Intrinsic Topological Quantization

Group $SO(10)$

Euler Class

Induced Topological Quantization

$G_{AB}, F_{+1234} = F_{+5678} = 2\mu, \phi = constant$

Group $U(4)$

Chern Class μ

Topological spectrum $\mu = \frac{2\pi n}{\phi}$

Supergravity IIB → **μ Quantized by Dirac**

Classical solution of fields

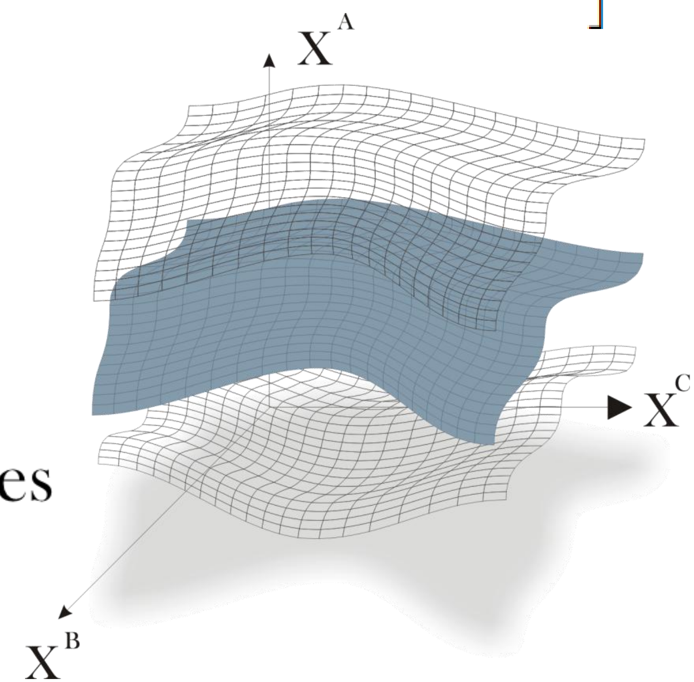
$$X^I = x_0^I \cos\mu\tau \quad + \frac{p_0^I}{\mu p^+} sen\mu\tau + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} \left[\alpha_n^I e^{\frac{-i}{\alpha' p^+}\omega_n\tau + n\sigma} + \right.$$

$$\left. + \tilde{\alpha}_n^I e^{\frac{-i}{\alpha' p^+}\omega_n\tau - n\sigma} + \alpha_n^{\dagger I} e^{\frac{+i}{\alpha' p^+}\omega_n\tau + n\sigma} + \tilde{\alpha}_n^{\dagger I} e^{\frac{+i}{\alpha' p^+}\omega_n\tau - n\sigma} \right]$$

$$\omega_n = \sqrt{n^2 + (\mu\alpha' p^+)^2}, \quad n \geq 0$$

ω_n is discretized by μ

Intrinsic topological quantization gives a topological spectrum for α and $\tilde{\alpha}$



Topological quantization. Induced metric h_{ab}

Induced metric $$h_{ab} = \frac{\partial X^A}{\partial x^a}\frac{\partial X^B}{\partial x^b} G_{AB}$$

$$G_{++}$$
$$G_{-+} = G_{+-}$$
$$G_{IJ} = \delta_{IJ}$$

$$h = \underbrace{\sum_{I=1}^{8} \partial_\sigma X^I \partial_\sigma X^I}_{f^2} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta}$$

Conformaly flat with f^2 as conformal factor

Group $SO(2)$ ➡ Euler class

$$\int e(R) = \iint \left(\partial_\sigma \left(\frac{\partial_\sigma f}{f} \right) - \partial_\tau \left(\frac{\partial_\tau f}{f} \right) \right) d\sigma\, d\tau$$

Already discretized n , μ y $\omega_n = \sqrt{n^2 + (\mu \alpha' p^+)^2}$ $\int e(R) = k \in \mathbb{Z}$

A particular case

Taking $\alpha_m = re^{-i\gamma}$, $\tilde{\alpha}_n = \tilde{r}e^{-i\tilde{\gamma}}$ and considering $X^I \in \mathbb{R}$

Fields:

$$X^1 = {x_0}^1 \sin(\mu\tau) + \frac{{p_0}^1}{\mu\, p^+}\cos(\mu\tau) + \sqrt{\frac{\alpha'}{2}}\frac{2r}{\sqrt{\omega_m}}\cos\left(\frac{\omega_m\tau + m\sigma}{\alpha' p^+} + \gamma\right)$$

$$X^2 = {x_0}^2 \sin(\mu\tau) + \frac{{p_0}^2}{\mu\, p^+}\cos(\mu\tau) + \sqrt{\frac{\alpha'}{2}}\frac{2\tilde{r}}{\sqrt{\omega_n}}\cos\left(\frac{\omega_n\tau - n\sigma}{\alpha' p^+} + \tilde{\gamma}\right)$$

Topological invariants

$\int e(R) = k \in \mathbb{Z}$

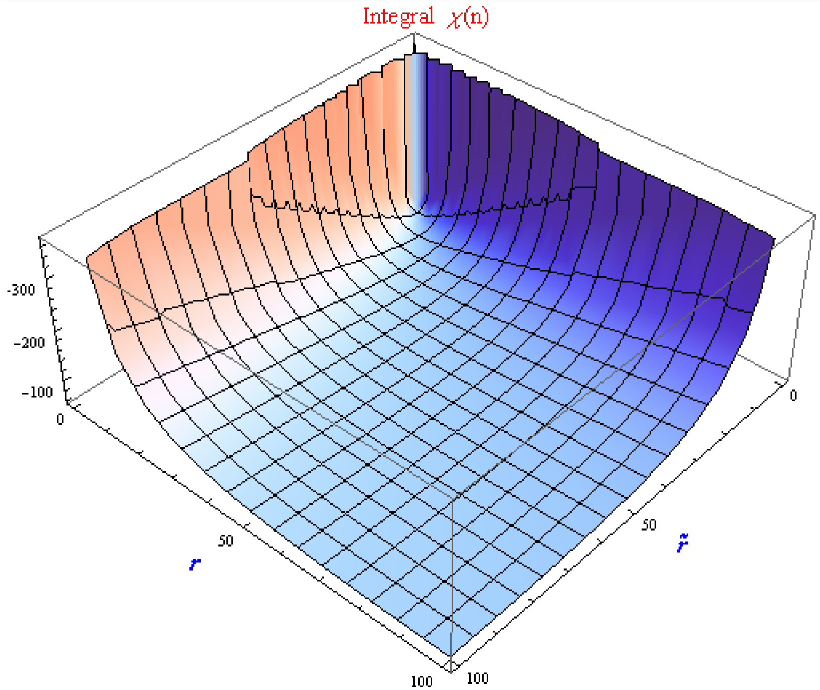
$$\int e(R)dxdy = \frac{({\omega_n}^2 - n^2)}{(n\omega_m + m\omega_n)} \frac{\sqrt{\omega_m}\, n}{\sqrt{\omega_n}\, m} \frac{\tilde{r}}{r} \qquad \int e(R)dydx = \frac{({\omega_m}^2 - m^2)}{(n\omega_m + m\omega_n)} \frac{\sqrt{\omega_n}\, m}{\sqrt{\omega_m}\, n} \frac{r}{\tilde{r}}$$

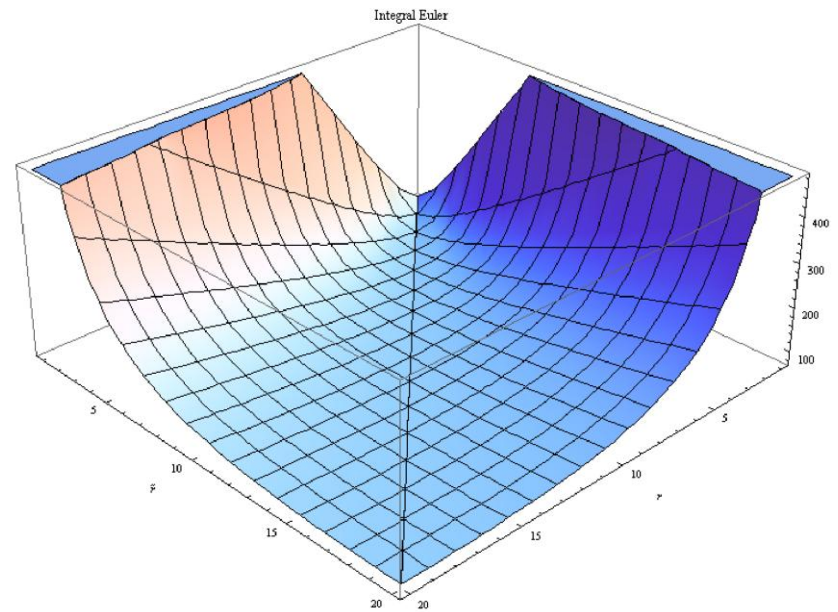
$$\Rightarrow \int e(R) = \frac{1}{2}\left(\int e(R)dxdy + \int e(R)dydx\right)$$

$$\Rightarrow \int e(R) = \frac{1}{2(n\omega_m + m\omega_n)\sqrt{\omega_m}\sqrt{\omega_n}nm}\left[\frac{\omega_m n^2\, \tilde{r}^2 + \omega_n m^2\, r^2}{r\tilde{r}}\right] = k \in \mathbb{Z}$$

Analytic integration

Numeric integration





$$\int e(R) = \frac{\pi\left(1-\omega^2\right)}{\omega}\frac{e^{i(\tilde{\gamma}-\gamma)}}{\tilde{\alpha}\alpha}\left(\tilde{\alpha}\tilde{\alpha}^{+} + \alpha\alpha^{+}\right)$$

Analyzing the topological spectrum

Hamiltonian of the system

Hamiltonian density: $\dot{X}^A \equiv \partial_\tau X^A$ $\qquad$ $\Pi_A = \dfrac{\partial \mathcal{L}}{\partial(\partial_\tau X^A)}$

$$\mathcal{H} = \dot{X}^A \Pi_A - \mathcal{L} = -\partial_\sigma X^A \partial_\sigma X^B G_{AB}$$

$$H = \int \mathcal{H}\, dxdy = \frac{\pi^2 \alpha'}{\omega_m \omega_n} \frac{(\omega_m n^2 \tilde{r}^2 + \omega_n m^2 r^2)}{(n\omega_m + m\omega_n)}$$

$$H = \frac{2\pi^2 nm}{\sqrt{\omega_m}\sqrt{\omega_n}} r\tilde{r}k$$

$$H = Ck \qquad k \in \mathbb{Z}$$

Energy discrete spectrum!

Conclusion

1. We found a discrete behavior of the energy for free massive bosonic fields using only topological considerations.
2. Topological quantization seem to be a subtle formalism to find a discrete behavior of physical systems.
3. There is a long road to walk...

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