The Geometry of Monopoles: New and Old I

H.W. Braden

Varna, June 2011

Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Overview

Equations	$\xrightarrow{Zero\ \ Curvature/Lax}$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

Overview

Equations	*Zero Curvature/Lax* $\longrightarrow$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

$t\mathbf{U} + \mathbf{C} \in \mathrm{Jac}(\mathcal{C})$

Overview

$$\begin{array}{ccc} \text{Equations} & \xrightarrow{\textit{Zero Curvature/Lax}} & \text{Spectral Curve } \mathcal{C} \subset \mathcal{S} \\ \uparrow & & \downarrow \\ \text{Reconstruction} & \longleftarrow & \text{Baker-Akhiezer Function} \end{array}$$

$t\mathbf{U} + \mathbf{C} \in \mathrm{Jac}(\mathcal{C})$

- BPS Monopoles
- Sigma Model reductions in AdS/CFT
- KP, KdV solitons
- Harmonic Maps
- SW Theory/Integrable Systems

Overview

Equations	*Zero Curvature/Lax* $\longrightarrow$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

$t\mathbf{U} + \mathbf{C} \in \mathrm{Jac}(\mathcal{C})$

- BPS Monopoles
- Sigma Model reductions in AdS/CFT
- KP, KdV solitons
- Harmonic Maps
- SW Theory/Integrable Systems

$\theta\left(t\mathbf{U} + \mathbf{C}|\tau\right)$

- Reduction of $F = *F$ (or static $V(\Phi) = 0$ with PS BC's)

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{ij}F^{ij} + \mathrm{Tr}\, D_i\Phi\, D^i\Phi + V(\Phi)$$

- $$B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = D_i\Phi$$
- A *monopole* of charge n

$$\left.\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

- Monopoles $\leftrightarrow$ Nahm Data $\leftrightarrow$ Hitchin Data

Three $n \times n$ matrices $T_i(s)$ with $s \in [0, 2]$ satisfying the following:

- N1 Nahm's equation $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k].$$
- N2 $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0, 2$. Residues form $su(2)$ irreducible n-dimensional representation.
- N3 $T_i(s) = -T_i^{\dagger}(s), \qquad T_i(s) = T_i^{t}(2 - s).$

BPS Monopoles

Nahm Data for charge n $SU(2)$ monopoles

Three $n \times n$ matrices $T_i(s)$ with $s \in [0, 2]$ satisfying the following:

N1 Nahm's equation $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k].$$

N2 $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0, 2$. Residues form $su(2)$ irreducible n-dimensional representation.

N3 $T_i(s) = -T_i^{\dagger}(s), \qquad T_i(s) = T_i^{t}(2 - s).$

$$A(\zeta) = T_1 + iT_2 - 2iT_3\zeta + (T_1 - iT_2)\zeta^2$$
$$M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$$

Nahm's eqn. $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k] \Longleftrightarrow [\frac{d}{ds} + M, A] = 0.$$

- $B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = D_i \Phi$
- Solve Weyl equation (charge n $SU(2)$ monopoles) $\mathbf{V}_{2n\times 2n}$

$$\Delta^\dagger \mathbf{V} = \left(1_{2n}\frac{d}{ds} + i\sum_{j=1}^{3} T_j(s)\otimes\sigma_j - \sum_{j=1}^{3} x_j 1_n \otimes \sigma_j \right) \mathbf{V}(\mathbf{x}, s) = 0$$

- Reconstruction $\mathbf{V}\boldsymbol{\mu} = (\mathbf{v}_1, \mathbf{v}_2)$, $\int_0^2 \mathbf{v}_a^\dagger(\mathbf{x}, s)\mathbf{v}_b(\mathbf{x}, s) ds = \delta_{ab}$

$$\Phi(\mathbf{x})_{ab} = i\int_0^2 s\mathbf{v}_a^\dagger(\mathbf{x}, s)\mathbf{v}_b(\mathbf{x}, s) ds, \qquad a, b = 1, 2$$

$$A_j(\mathbf{x})_{ab} = i\int_0^2 \mathbf{v}_a^\dagger(\mathbf{x}, s)\frac{\partial}{\partial x_j}\mathbf{v}_b(\mathbf{x}, s) ds, \quad i = 1, 2, 3$$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

  $$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

$$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ for $H_1(\mathcal{C}, \mathbb{Z})$

$$\mathfrak{a}_i \cap \mathfrak{b}_j = -\mathfrak{b}_j \cap \mathfrak{a}_i = \delta_{ij}$$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

$$P(\eta,\zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ for $H_1(\mathcal{C}, \mathbb{Z})$

$$\mathfrak{a}_i \cap \mathfrak{b}_j = -\mathfrak{b}_j \cap \mathfrak{a}_i = \delta_{ij}$$

- Holomorphic differentials du_i $(i = 1, \ldots, g)$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}:\ 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

$$P(\eta,\zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ for $H_1(\mathcal{C}, \mathbb{Z})$

$$\mathfrak{a}_i \cap \mathfrak{b}_j = -\mathfrak{b}_j \cap \mathfrak{a}_i = \delta_{ij}$$

- Holomorphic differentials du_i $(i = 1, \ldots, g)$
- Period Matrix $\tau = \mathcal{B}\mathcal{A}^{-1}$ where

$$\Pi := \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \oint_{\mathfrak{a}_i} du_j \\ \oint_{\mathfrak{b}_i} du_j \end{pmatrix}$$

- $[\frac{d}{ds}+M(\zeta),A(\zeta)]=0,\quad \mathcal{C}:\ 0=\det(\eta 1_n+A(\zeta)):=P(\eta,\zeta)$

$$P(\eta,\zeta)=\eta^n+a_1(\zeta)\eta^{n-1}+\ldots+a_n(\zeta),\qquad \deg a_r(\zeta)\leq 2r$$

- Homology basis $\{\gamma_i\}_{i=1}^{2g}=\{\mathfrak{a}_i,\mathfrak{b}_i\}_{i=1}^{g}$ for $H_1(\mathcal{C},\mathbb{Z})$

$$\mathfrak{a}_i\cap\mathfrak{b}_j=-\mathfrak{b}_j\cap\mathfrak{a}_i=\delta_{ij}$$

- Holomorphic differentials du_i $(i=1,\ldots,g)$
- Period Matrix $\tau=\mathcal{B}\mathcal{A}^{-1}$ where

$$\Pi:=\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix}=\begin{pmatrix}\oint_{\mathfrak{a}_i}du_j\\ \oint_{\mathfrak{b}_i}du_j\end{pmatrix}$$

- normalized holomorphic differentials ω_i, $\oint_{\mathfrak{a}_i}\omega_j=\delta_{ij}$ $\oint_{\mathfrak{b}_i}\omega_j=\tau_{ij}$

- $[\frac{d}{ds} + M(\zeta), A(\zeta)] = 0, \quad \mathcal{C}: \; 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

$$P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ for $H_1(\mathcal{C}, \mathbb{Z})$

$$\mathfrak{a}_i \cap \mathfrak{b}_j = -\mathfrak{b}_j \cap \mathfrak{a}_i = \delta_{ij}$$

- Holomorphic differentials du_i $(i = 1, \ldots, g)$
- Period Matrix $\tau = \mathcal{B}\mathcal{A}^{-1}$ where

$$\Pi := \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \oint_{\mathfrak{a}_i} du_j \\ \oint_{\mathfrak{b}_i} du_j \end{pmatrix}$$

- normalized holomorphic differentials ω_i, $\oint_{\mathfrak{a}_i} \omega_j = \delta_{ij}$ $\oint_{\mathfrak{b}_i} \omega_j = \tau_{ij}$
- genus given by Riemann Hurwitz formula $g_{\text{monopole}} = (n-1)^2$

Baker-Akhiezer Function

Existence: Krichever's Theorem (1977)

Let $\mathcal{C}$ be a smooth algebraic curve of genus $g_{\mathcal{C}}$ with $n \geqslant 1$ punctures P_j, $j = 1, \ldots, n$. Then for each set of $g_{\mathcal{C}} + n - 1$ points $\delta_1, \ldots, \delta_{g_{\mathcal{C}}+n-1}$ in general position, there exists a unique function $\Psi_j\,(t, P)$ and local coordinates $w_j(P)$ for which $w_j(P_j) = 0$, such that

1. The function Ψ_j of $P \in \mathcal{C}$ is meromorphic outside the punctures and has at most simple poles at δ_r (if all of them are distinct);
2. In the neighbourhood of the puncture P_l the function Ψ_j has the form (for $i \in \mathbb{N}^+$, $w_l = w_l\,(P)$)

$$\Psi_j\,(s, P) = e^{s\, {w_l}^{-i}} \left(\delta_{jl} + \sum_{k=1}^{\infty} \alpha_{jl}^{k}\,(s)\, w_l^k \right)$$

Meromorphic differential describe flows

$$w_j(P_j) = 0 \qquad d\Omega^{(i)} = d\left(w_j^{-i} + 0\,(w_j)\right) \qquad \oint_{\mathfrak{a}_k} d\Omega^{(i)} = 0$$

Reconstruction

- Reconstruct $A(\zeta)$ in terms of its joint eigenfunctions

$$(\eta 1_n + A(\zeta))\,\hat{w} = 0$$
$$\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$$

Reconstruction

- Reconstruct $A(\zeta)$ in terms of its joint eigenfunctions

$$(\eta 1_n + A(\zeta))\,\hat{w} = 0$$
$$\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$$

- Solve $\hat{w}$ in terms of Baker-Akhiezer function.

Reconstruction

- Reconstruct $A(\zeta)$ in terms of its joint eigenfunctions

$$(\eta 1_n + A(\zeta))\,\hat{w} = 0$$
$$\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$$

- Solve $\hat{w}$ in terms of Baker-Akhiezer function.
- $M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$ poles at $\zeta = \infty$

$$\frac{P(\eta,\zeta)}{\zeta^{2n}} \sim \prod_{j=1}^{n}\left(\frac{\eta}{\zeta^2} - \rho_j\right) \qquad \frac{\eta}{\zeta} = \rho_j\zeta,\ \zeta \sim \infty_j \ \textit{n} \text{ points on } \mathcal{C}$$

$$d\left(\frac{\eta}{\zeta}\right) = \left(-\frac{\rho_j}{t^2} + O(1)\right)dt \qquad \zeta = \frac{1}{t} \sim \infty_j$$

Reconstruction

- Reconstruct $A(\zeta)$ in terms of its joint eigenfunctions

$$(\eta 1_n + A(\zeta))\,\hat{w} = 0$$
$$\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$$

- Solve $\hat{w}$ in terms of Baker-Akhiezer function.
- $M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$ poles at $\zeta = \infty$

$$\frac{P(\eta,\zeta)}{\zeta^{2n}} \sim \prod_{j=1}^{n}\left(\frac{\eta}{\zeta^2} - \rho_j\right) \qquad \frac{\eta}{\zeta} = \rho_j\zeta,\ \zeta \sim \infty_j \text{ } n \text{ points on } \mathcal{C}$$

$$d\left(\frac{\eta}{\zeta}\right) = \left(-\frac{\rho_j}{t^2} + O(1)\right)dt \qquad \zeta = \frac{1}{t} \sim \infty_j$$

- $\exists!$ meromorphic differential $\gamma_\infty \equiv \gamma_\infty(P) = \left(\frac{\rho_j}{t^2} + O(1)\right)\mathrm{d}t$,

as $P \to \infty_j$, $\oint_{\mathfrak{a}_k} \gamma_\infty(P) = 0$, $\quad \mathbf{U} = \frac{1}{2i\pi}\oint_{\mathfrak{b}} \gamma_\infty$

Reconstruction

- Ercolani and Sinha (1979) solve gauge transform of $\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$ in terms of Baker-Akhiezer function.

Reconstruction

- Ercolani and Sinha (1979) solve gauge transform of $\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$ in terms of Baker-Akhiezer function.
- BE (2007) $\mathbf{y} = \left(\frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2}, -\zeta\right) \qquad \widehat{\mathbf{u}} = \widehat{\mathbf{u}}(\zeta) := \imath\,\frac{\mathbf{y}\times\overline{\mathbf{y}}}{\mathbf{y}\cdot\overline{\mathbf{y}}}$

Reconstruction

- Ercolani and Sinha (1979) solve gauge transform of $\left(\frac{d}{ds}+M(\zeta)\right)\hat{w}=0$ in terms of Baker-Akhiezer function.
- BE (2007) $\mathbf{y}=\left(\frac{1+\zeta^2}{2\imath},\frac{1-\zeta^2}{2},-\zeta\right) \quad \widehat{\mathbf{u}}=\widehat{\mathbf{u}}(\zeta):=\imath\,\frac{\mathbf{y}\times\overline{\mathbf{y}}}{\mathbf{y}\cdot\overline{\mathbf{y}}}$
- $\mathbf{w}(P)=(1_2+\widehat{\mathbf{u}}\cdot\boldsymbol{\sigma})\,e^{-\imath s[(x_1-\imath x_2)\zeta-\imath x_3]}|\chi>\otimes\widehat{\mathbf{w}}(P)$
  Then $\Delta\mathbf{w}=0$ with $\eta=2\mathbf{y}\cdot\mathbf{x}$.

Reconstruction

- Ercolani and Sinha (1979) solve gauge transform of $\left(\frac{d}{ds} + M(\zeta)\right)\hat{w} = 0$ in terms of Baker-Akhiezer function.
- BE (2007) $\mathbf{y} = \left(\frac{1+\zeta^2}{2\imath}, \frac{1-\zeta^2}{2}, -\zeta\right) \quad \widehat{\mathbf{u}} = \widehat{\mathbf{u}}(\zeta) := \imath\, \frac{\mathbf{y}\times\overline{\mathbf{y}}}{\mathbf{y}\cdot\overline{\mathbf{y}}}$
- $\mathbf{w}(P) = (1_2 + \widehat{\mathbf{u}}\cdot\boldsymbol{\sigma})\, e^{-\imath s[(x_1-\imath x_2)\zeta - \imath x_3]}|\chi > \otimes\, \widehat{\mathbf{w}}(P)$
  Then $\Delta\mathbf{w} = 0$ with $\eta = 2\mathbf{y}\cdot\mathbf{x}$.
- 1. Given $\mathbf{x}$ solve $\det(2\mathbf{y}\cdot\mathbf{x} + A(\zeta)) = 0$ of degree $2n$ in ζ
  2. Let P_i be the corresponding $2n$ points on $\mathcal{C}$.
  3. Construct the $2n\times 2n$ matrix $\mathbf{W} = (\mathbf{w}(P_i))$
  4. $\mathbf{V} = (\mathbf{W}^\dagger)^{-1}, \quad \Delta^\dagger\mathbf{V} = 0$ Problem: extract norm. solns.

Reconstruction

- Ercolani and Sinha (1979) solve gauge transform of $\left(\frac{d}{ds}+M(\zeta)\right)\hat{w}=0$ in terms of Baker-Akhiezer function.
- BE (2007) $\mathbf{y}=\left(\frac{1+\zeta^2}{2\imath},\frac{1-\zeta^2}{2},-\zeta\right)$ $\quad \widehat{\mathbf{u}}=\widehat{\mathbf{u}}(\zeta):=\imath\,\frac{\mathbf{y}\times\overline{\mathbf{y}}}{\mathbf{y}\cdot\overline{\mathbf{y}}}$
- $\mathbf{w}(P)=(1_2+\widehat{\mathbf{u}}\cdot\boldsymbol{\sigma})\,e^{-\imath s[(x_1-\imath x_2)\zeta-\imath x_3]}|\chi>\otimes\widehat{\mathbf{w}}(P)$
  Then $\Delta\mathbf{w}=0$ with $\eta=2\mathbf{y}\cdot\mathbf{x}$.
-
  1. Given $\mathbf{x}$ solve $\det(2\mathbf{y}\cdot\mathbf{x}+A(\zeta))=0$ of degree $2n$ in ζ
  2. Let P_i be the corresponding $2n$ points on $\mathcal{C}$.
  3. Construct the $2n\times 2n$ matrix $\mathbf{W}=(\mathbf{w}(P_i))$
  4. $\mathbf{V}=(\mathbf{W}^\dagger)^{-1}$, $\quad \Delta^\dagger\mathbf{V}=0$ Problem: extract norm. solns.
- Panagopoulos (1983): Integrals computed in closed form

$$\int \mathbf{v}_a^\dagger(\mathbf{x},s)\mathbf{v}_b(\mathbf{x},s)\mathrm{d}s=\mathbf{v}_a^\dagger(\mathbf{x},s)\mathcal{F}^{-1}(\mathbf{x},s)\mathbf{v}_b(\mathbf{x},s)$$

$$\mathcal{F}(\mathbf{x},s)=\frac{1}{r^2}\mathcal{H}\mathcal{T}\mathcal{H}-\mathcal{T},\mathcal{T}=\sum_{i=1}^{3}\sigma_i\otimes T_i(s),\mathcal{H}=\sum_{i=1}^{3}x_i\sigma_i\otimes 1_n.$$

- **Abel Map** $\phi_*(P) : \mathcal{C} \to \mathbb{C}^g$, $\phi_*(P) = \left(\int_*^P \omega_1, \ldots, \int_*^P \omega_g \right)$
  $\Lambda \subset \mathbb{C}^g$: $\quad \mathbb{Z}^g \oplus \mathbb{Z}^g \tau \quad \mathbb{C}^g / \Lambda \equiv \mathrm{Jac}(\mathcal{C})$

- **Abel Map** $\phi_*(P) : \mathcal{C} \to \mathbb{C}^g$, $\phi_*(P) = \left(\int_*^P \omega_1, \ldots, \int_*^P \omega_g \right)$

  $\Lambda \subset \mathbb{C}^g$: $\quad \mathbb{Z}^g \oplus \mathbb{Z}^g \tau \quad \mathbb{C}^g / \Lambda \equiv \mathrm{Jac}(\mathcal{C})$

- Abel $\quad \int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_g} \omega = \mathbf{z} \quad \phi_{P_0}\left(\sum_{i=1}^{g} P_i \right) = \mathbf{z}$

- **Abel Map** $\phi_*(P) : \mathcal{C} \to \mathbb{C}^g$, $\phi_*(P) = \left(\int_*^P \omega_1, \ldots, \int_*^P \omega_g \right)$

  $\Lambda \subset \mathbb{C}^g$: $\quad \mathbb{Z}^g \oplus \mathbb{Z}^g \tau \quad \mathbb{C}^g/\Lambda \equiv \mathrm{Jac}(\mathcal{C})$
- Abel $\quad \int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_g} \omega = \mathbf{z} \quad \phi_{P_0}\left(\sum_{i=1}^{g} P_i \right) = \mathbf{z}$
- Clebsch and Gordan (1886). Let Ω_{P_+,P_-} meromorphic differentials of the third kind with simple poles at $P_\pm$ and having residues ± 1. Suppose $X_1, Y_1, \ldots, X_s, Y_s$ are distinct pairs of points on $\mathcal{C}$. For $i = 1, \ldots s$ we may solve

$$\int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+s}} \omega = \mathbf{z}, \qquad \int_{P_0}^{P_1} \Omega_{X_i,Y_i} + \cdots + \int_{P_0}^{P_{g+s}} \Omega_{X_i,Y_i} = Z_i$$

- Braden and Fedorov

$$\int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+n-1}} \omega = \mathbf{z}, \qquad \int_{P_0}^{P_1} \Omega_{j1} + \cdots + \int_{P_0}^{P_{g+n-1}} \Omega_{j1} = Z_j$$

Reconstruction

Theta Functions

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \operatorname{Im}\tau > 0$

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \operatorname{Im}\tau > 0$
- $\theta(\mathbf{z}+\mathbf{p}\,;\tau) = \theta(\mathbf{z};\tau)$
  $\theta(\mathbf{z}+\mathbf{p}\tau;\tau) = \exp\{-i\pi(\mathbf{p}^T\tau\mathbf{p} + 2\mathbf{z}^T\mathbf{p})\}\,\theta(\mathbf{z};\tau), \qquad \mathbf{p}\in\mathbb{Z}^g$

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \mathrm{Im}\,\tau > 0$
- $\theta(\mathbf{z}+\mathbf{p}\,;\tau) = \theta(\mathbf{z};\tau)$
  $\theta(\mathbf{z}+\mathbf{p}\tau;\tau) = \exp\{-i\pi(\mathbf{p}^T\tau\mathbf{p} + 2\mathbf{z}^T\mathbf{p})\}\,\theta(\mathbf{z};\tau), \qquad \mathbf{p}\in\mathbb{Z}^g$
- $\theta(e\,|\,\tau) = 0 \iff e \in \Theta \subset \mathrm{Jac}(\mathcal{C})$
- **Riemann's Theorem** $e \equiv \phi_Q\left(\sum_{i=1}^{g-1} P_i\right) + K_Q \quad \phi(\mathcal{C}^{g-1})+K_Q=\Theta$

Reconstruction

Theta Functions

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \mathrm{Im}\,\tau > 0$
- $\theta(\mathbf{z}+\mathbf{p}\,;\tau) = \theta(\mathbf{z};\tau)$
  $\theta(\mathbf{z}+\mathbf{p}\tau;\tau) = \exp\{-i\pi(\mathbf{p}^T\tau\mathbf{p} + 2\mathbf{z}^T\mathbf{p})\}\,\theta(\mathbf{z};\tau), \qquad \mathbf{p}\in\mathbb{Z}^g$
- $\theta(e\,|\,\tau) = 0 \Longleftrightarrow e \in \Theta \subset \mathrm{Jac}(\mathcal{C})$
- **Riemann's Theorem** $e \equiv \phi_Q\left(\sum_{i=1}^{g-1} P_i\right) + K_Q \quad \phi(\mathcal{C}^{g-1}) + K_Q = \Theta$

$$\mathrm{mult}_{\mathrm{e}}\,\theta = \mathrm{i}\left(\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}\right) = \dim \mathrm{H}^1(\mathcal{C}, \mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}}) = \dim \mathrm{H}^0(\mathcal{C}, \mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}})$$

Reconstruction

Theta Functions

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \operatorname{Im}\tau > 0$
- $\theta(\mathbf{z}+\mathbf{p}\,;\tau) = \theta(\mathbf{z};\tau)$
  $\theta(\mathbf{z}+\mathbf{p}\tau;\tau) = \exp\{-i\pi(\mathbf{p}^T\tau\mathbf{p} + 2\mathbf{z}^T\mathbf{p})\}\,\theta(\mathbf{z};\tau), \qquad \mathbf{p}\in\mathbb{Z}^g$
- $\theta(e\,|\,\tau) = 0 \iff e \in \Theta \subset \mathrm{Jac}(\mathcal{C})$
- **Riemann's Theorem** $e \equiv \phi_Q\left(\sum_{i=1}^{g-1} P_i\right) + K_Q \quad \phi(\mathcal{C}^{g-1})+K_Q=\Theta$

$$\mathrm{mult_e}\,\theta = \mathrm{i}\left(\sum_{\mathrm{i=1}}^{\mathrm{g-1}} \mathrm{P_i}\right) = \dim \mathrm{H}^1(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i=1}}^{\mathrm{g-1}}\mathrm{P_i}}) = \dim \mathrm{H}^0(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i=1}}^{\mathrm{g-1}}\mathrm{P_i}})$$

- There are very efficient evaluations of θ

- $\theta(\mathbf{z};\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp(i\pi\mathbf{n}^T\tau\mathbf{n} + 2i\pi\mathbf{z}^T\mathbf{n}), \qquad \mathrm{Im}\,\tau > 0$
- $\theta(\mathbf{z}+\mathbf{p}\,;\tau) = \theta(\mathbf{z};\tau)$
  $\theta(\mathbf{z}+\mathbf{p}\tau;\tau) = \exp\{-i\pi(\mathbf{p}^T\tau\mathbf{p} + 2\mathbf{z}^T\mathbf{p})\}\,\theta(\mathbf{z};\tau), \qquad \mathbf{p}\in\mathbb{Z}^g$
- $\theta(e\,|\,\tau) = 0 \iff e \in \Theta \subset \mathrm{Jac}(\mathcal{C})$
- **Riemann's Theorem** $e \equiv \phi_Q\left(\sum_{i=1}^{g-1} P_i\right) + K_Q \quad \phi(\mathcal{C}^{g-1})+K_Q=\Theta$

$$\mathrm{mult}_e\,\theta = \mathrm{i}\left(\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}\right) = \dim \mathrm{H}^1(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}}) = \dim \mathrm{H}^0(\mathcal{C},\mathcal{L}_{\sum_{\mathrm{i}=1}^{\mathrm{g}-1}\mathrm{P_i}})$$

- There are very efficient evaluations of θ
- Functions $\sum_{i=1}^N \phi(R_i) = \sum_{i=1}^N \phi(S_i) + n + m\cdot\tau$

$$f(P) = e^{2\pi i\int_{P_0}^P m\cdot\omega}\prod_{i=1}^N \frac{\theta(\phi(P)-\phi(R_i)-K;\tau)}{\theta(\phi(P)-\phi(S_i)-K;\tau)}$$

Reconstruction

Divisors, Line bundles, Θ

- Line bundles $\longleftrightarrow$ Divisors $\delta = \sum_i n_i P_i$, $n_i \in \mathbb{Z}$, $P_i \in \mathcal{C}$.
  - effective $n_i \geq 0\ \forall i$. $\quad \delta$ effective $\iff \dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1$.
  - nonspecial $\iff \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0$

- Line bundles $\longleftrightarrow$ Divisors $\delta = \sum_i n_i P_i$, $n_i \in \mathbb{Z}$, $P_i \in \mathcal{C}$.
  - effective $n_i \geq 0\ \forall i$. δ effective $\Longleftrightarrow \dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1$.
  - nonspecial $\Longleftrightarrow \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0$
- Riemann-Roch
  $\dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \deg L_\delta + 1 - g_{\mathcal{C}} + \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta))$

- Line bundles $\longleftrightarrow$ Divisors $\delta = \sum_i n_i P_i$, $n_i \in \mathbb{Z}$, $P_i \in \mathcal{C}$.
  - effective $n_i \geq 0\ \forall i$. δ effective $\iff \dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1$.
  - nonspecial $\iff \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0$
- Riemann-Roch
  $\dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \deg L_\delta + 1 - g_{\mathcal{C}} + \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta))$
- Example: $\deg L_\delta = g_{\mathcal{C}} - 1$

$$\dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1 \iff \delta \text{ effective}$$

$$\iff \delta \in \Theta$$

$$\mathrm{multiplicity}_L\, \theta = \dim H^0(\mathcal{C}, \mathcal{O}(L))$$

- Line bundles $\longleftrightarrow$ Divisors $\delta = \sum_i n_i P_i$, $n_i \in \mathbb{Z}$, $P_i \in \mathcal{C}$.
  - effective $n_i \geq 0\ \forall i$. δ effective $\Longleftrightarrow \dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1$.
  - nonspecial $\Longleftrightarrow \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0$
- Riemann-Roch
  $\dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \deg L_\delta + 1 - g_\mathcal{C} + \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta))$
- Example: $\deg L_\delta = g_\mathcal{C} - 1$

$$\dim H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \dim H^1(\mathcal{C}, \mathcal{O}(L_\delta)) \geq 1 \Longleftrightarrow \delta \text{ effective}$$

$$\Longleftrightarrow \delta \in \Theta$$

$$\text{multiplicity}_L\, \theta = \dim H^0(\mathcal{C}, \mathcal{O}(L))$$

- in general position $\Longleftrightarrow \displaystyle\sum_{s=1}^{g_\mathcal{C}+n-1} \phi(\delta_s) - \sum_{l=1}^{n} \phi(P_l)$ noneffective

$$\Longleftrightarrow \sum_{s=1}^{g_\mathcal{C}+n-1} \phi(\delta_s) - \sum_{l=1}^{n} \phi(P_l) + K_Q \notin \Theta$$

- Baker-Akhiezer function

$$\Psi_j(s,P)=g_j(P)\,\frac{\theta(\phi(P)+s\mathbf{U}-\zeta_j)}{\theta(\phi(P)-\zeta_j)}\,e^{s\int_{P_0}^{P}\gamma_\infty}\times\frac{\theta(\phi(P_j)-\zeta_j)}{\theta(\phi(P_j)+s\mathbf{U}-\zeta_j)}$$

$$\mathbf{U}=\frac{1}{2\pi i}\oint_{\mathfrak{b}}\gamma_\infty,\qquad g_j(P_k)=\delta_{jk}$$

$$\zeta_j=\sum_{s=1}^{g_{\mathcal{C}}+n-1}\phi(\delta_s)-\sum_{\substack{l=1\\ l\neq j}}^{n}\phi(P_l)+K$$

- Baker-Akhiezer function

$$\Psi_j(s,P) = g_j(P)\,\frac{\theta(\phi(P)+s\mathbf{U}-\zeta_j)}{\theta(\phi(P)-\zeta_j)}\,e^{s\int_{P_0}^{P}\gamma_\infty} \times \frac{\theta(\phi(P_j)-\zeta_j)}{\theta(\phi(P_j)+s\mathbf{U}-\zeta_j)}$$

$$\mathbf{U} = \frac{1}{2\pi i}\oint_{\mathfrak{b}}\gamma_\infty, \qquad g_j(P_k) = \delta_{jk}$$

$$\zeta_j = \sum_{s=1}^{g_{\mathcal{C}}+n-1}\phi(\delta_s) - \sum_{\substack{l=1\\ l\neq j}}^{n}\phi(P_l) + K$$

- in general position: 1) $\theta(\phi(P_j)-\zeta_j)\neq 0$, 2) $\exists!\quad g_j(P)$

Reconstruction

Functions in terms of theta functions

- Baker-Akhiezer function

$$\Psi_j\,(s,P) = g_j(P)\,\frac{\theta(\phi(P)+s\mathbf{U}-\zeta_j)}{\theta(\phi(P)-\zeta_j)}\,e^{s\int_{P_0}^{P}\gamma_\infty} \times \frac{\theta(\phi(P_j)-\zeta_j)}{\theta(\phi(P_j)+s\mathbf{U}-\zeta_j)}$$

$$\mathbf{U} = \frac{1}{2\pi i}\oint_{\mathfrak{b}} \gamma_\infty, \qquad g_j(P_k) = \delta_{jk}$$

$$\zeta_j = \sum_{s=1}^{g_{\mathcal{C}}+n-1} \phi(\delta_s) - \sum_{\substack{l=1\\ l\neq j}}^{n} \phi(P_l) + K$$

- in general position: 1) $\theta(\phi(P_j)-\zeta_j) \neq 0$, 2) $\exists! \quad g_j(P)$
- Danger $\phi(P_j) + s\mathbf{U} - \zeta_j \in \Theta$

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

H2 $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$.
$\mathcal{L}^\lambda := \mathcal{L}^\lambda(0), \qquad \mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^*\mathcal{O}(m)$
$\mathcal{L}^2$ is trivial on $\mathcal{C}$ and $\mathcal{L}^1(n-1)$ is real.

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

H2 $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$.
$\mathcal{L}^\lambda := \mathcal{L}^\lambda(0), \qquad \mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^* \mathcal{O}(m)$
$\mathcal{L}^2$ is trivial on $\mathcal{C}$ and $\mathcal{L}^1(n-1)$ is real.
$\mathcal{L}^2$ is trivial $\implies \exists$ nowhere-vanishing holomorphic section.

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

H2 $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp\left(-\lambda\eta/\zeta\right)$.
$\mathcal{L}^\lambda := \mathcal{L}^\lambda(0), \qquad \mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^*\mathcal{O}(m)$
$\mathcal{L}^2$ is trivial on $\mathcal{C}$ and $\mathcal{L}^1(n-1)$ is real.
$\mathcal{L}^2$ is trivial $\implies$ $\exists$ nowhere-vanishing holomorphic section.

H3 $H^0(\mathcal{C}, \mathcal{L}^s(n-2)) = 0$ for $s \in (0,2) \Longleftrightarrow \theta(s\mathbf{U} + \mathbf{C} \,|\, \tau) \neq 0$

$$\mathbf{C} = K_Q + \phi_Q\left((n-2)\sum_{k=1}^{n} \infty_k\right)$$

H1 $\mathcal{C} \subset T\mathbb{P}^1$ Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

H2 $\mathcal{L}^\lambda(m)$ the holomorphic line bundle on $T\mathbb{P}^1$ with transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$.
$\mathcal{L}^\lambda := \mathcal{L}^\lambda(0), \qquad \mathcal{L}^\lambda(m) \equiv \mathcal{L}^\lambda \otimes \pi^*\mathcal{O}(m)$
$\mathcal{L}^2$ is trivial on $\mathcal{C}$ and $\mathcal{L}^1(n-1)$ is real.
$\mathcal{L}^2$ is trivial $\implies \exists$ nowhere-vanishing holomorphic section.

H3 $H^0(\mathcal{C}, \mathcal{L}^s(n-2)) = 0$ for $s \in (0,2) \Longleftrightarrow \theta(s\mathbf{U} + \mathbf{C} \,|\, \tau) \neq 0$

$$\mathbf{C} = K_Q + \phi_Q\left((n-2)\sum_{k=1}^{n} \infty_k\right)$$

$H^0(\mathcal{C}, \mathcal{O}(\mathcal{L}^s(n-2))) = 0 \Rightarrow H^0(\mathcal{C}, \mathcal{O}(\mathcal{L}^s)) = 0, \; s \in (0,2)$.
$\mathcal{O}(\mathcal{L}^s) \hookrightarrow \mathcal{O}(\mathcal{L}^s(n-2)) \times$ a section of $\pi^*\mathcal{O}(n-2)|_{\mathcal{C}}$

The Ercolani-Sinha Constraints

- $\mathcal{L}^2$ trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

  $\mathrm{dlog}\, f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1,\quad \exp\oint_\gamma \mathrm{dlog}\, f_0 = 1 \quad \forall\gamma\in H_1(\mathbb{Z},\mathcal{C})$

The Ercolani-Sinha Constraints

- $\mathcal{L}^2$ trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

$$\operatorname{dlog} f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \operatorname{dlog} f_1, \quad \exp \oint_\gamma \operatorname{dlog} f_0 = 1 \quad \forall \gamma \in H_1(\mathbb{Z}, \mathcal{C})$$

- $\gamma_\infty(P) = \frac{1}{2}\operatorname{dlog} f_0(P) + \imath\pi \sum_{j=1}^{g} m_j\,\omega_j(P), \qquad \oint_{\mathfrak{a}_k} \omega_j = \delta_{jk}$

$$\oint_{\mathfrak{b}_k} \gamma_\infty = \imath\pi n_k + \imath\pi \sum_{l=1}^{g} m_l \tau_{lk}, \qquad 2\mathbf{U} \in \Lambda$$

The Ercolani-Sinha Constraints

- $\mathcal{L}^2$ trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

  $$\mathrm{dlog}\, f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1, \quad \exp\oint_\gamma \mathrm{dlog}\, f_0 = 1 \quad \forall \gamma \in H_1(\mathbb{Z},\mathcal{C})$$

- $$\gamma_\infty(P) = \frac{1}{2}\mathrm{dlog}\, f_0(P) + \imath\pi\sum_{j=1}^{g} m_j\,\omega_j(P), \qquad \oint_{\mathfrak{a}_k}\omega_j = \delta_{jk}$$

  $$\oint_{\mathfrak{b}_k}\gamma_\infty = \imath\pi n_k + \imath\pi\sum_{l=1}^{g} m_l\tau_{lk}, \qquad 2\mathbf{U}\in\Lambda$$

- Ercolani-Sinha Constraints: The following are equivalent:
  1. $\mathcal{L}^2$ is trivial on $\mathcal{C}$.
  2. $2\mathbf{U}\in\Lambda \Longleftrightarrow \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty,\ldots,\oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}$.
  3. $\exists$ 1-cycle $\mathfrak{es} = \mathbf{n}\cdot\mathfrak{a} + \mathbf{m}\cdot\mathfrak{b}$ s.t. for every holomorphic differential
     $$\Omega = \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\,d\zeta, \quad \oint_{\mathfrak{es}}\Omega = -2\beta_0$$

The Ercolani-Sinha Constraints

- $\mathcal{L}^2$ trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

$$\mathrm{dlog}\, f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1, \quad \exp \oint_\gamma \mathrm{dlog}\, f_0 = 1 \quad \forall \gamma \in H_1(\mathbb{Z},\mathcal{C})$$

- $\gamma_\infty(P) = \frac{1}{2}\mathrm{dlog}\, f_0(P) + \imath\pi \sum_{j=1}^{g} m_j \omega_j(P), \qquad \oint_{\mathfrak{a}_k} \omega_j = \delta_{jk}$

$$\oint_{\mathfrak{b}_k} \gamma_\infty = \imath\pi n_k + \imath\pi \sum_{l=1}^{g} m_l \tau_{lk}, \qquad 2\mathbf{U} \in \Lambda$$

- Ercolani-Sinha Constraints: The following are equivalent:
  1. $\mathcal{L}^2$ is trivial on $\mathcal{C}$.
  2. $2\mathbf{U} \in \Lambda \Longleftrightarrow \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}$.
  3. $\exists$ 1-cycle $\mathfrak{es} = \mathbf{n}\cdot\mathfrak{a} + \mathbf{m}\cdot\mathfrak{b}$ s.t. for every holomorphic differential
     $\Omega = \dfrac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\, d\zeta, \ \oint_{\mathfrak{es}} \Omega = -2\beta_0$
- **H3** $\mathcal{L}^s$ trivial $\Longleftrightarrow s\,\mathbf{U} \in \Lambda, \qquad 2\mathbf{U}$ is a primitive vector in Λ

Summary

- Lax Pair $[\frac{d}{ds} + M(\zeta), L(\zeta)] = 0$ leads to the study of a curve

$$\mathcal{C}: \; 0 = \det(\eta 1_n + L(\zeta)) := P(\eta, \zeta)$$

Summary

- Lax Pair $[\frac{d}{ds} + M(\zeta), L(\zeta)] = 0$ leads to the study of a curve
$$\mathcal{C}: \ 0 = \det(\eta 1_n + L(\zeta)) := P(\eta, \zeta)$$
- The flows (via M) are governed by meromorphic differentials γ_∞ on $\mathcal{C}$.
$$\mathbf{U} = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}} \gamma_\infty$$

Summary

- Lax Pair $[\frac{d}{ds} + M(\zeta), L(\zeta)] = 0$ leads to the study of a curve
$$\mathcal{C}: \ 0 = \det(\eta 1_n + L(\zeta)) := P(\eta, \zeta)$$
- The flows (via M) are governed by meromorphic differentials γ_∞ on $\mathcal{C}$.
$$\mathbf{U} = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}} \gamma_\infty$$
- The solution constructed via $\theta\left(s\mathbf{U} + \mathbf{C}|\tau\right)$

Summary

- Lax Pair $[\frac{d}{ds} + M(\zeta), L(\zeta)] = 0$ leads to the study of a curve

$$\mathcal{C}: \; 0 = \det(\eta 1_n + L(\zeta)) := P(\eta, \zeta)$$

- The flows (via M) are governed by meromorphic differentials γ_∞ on $\mathcal{C}$.

$$\mathbf{U} = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}} \gamma_\infty$$

- The solution constructed via $\theta\left(s\mathbf{U} + \mathbf{C}|\tau\right)$
- Transcendental constraints.
  1. Flows and Theta Divisor. $s\mathbf{U} + \mathbf{C} \notin \Theta$
  2. $\mathcal{L}^2$ trivial $\iff 2\mathbf{U} \in \Lambda \iff \mathbf{U} = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}$
     ES conditions impose g *transcendental constraints* on curve

$$\sum_{j=2}^{n}(2j+1) - g = (n+3)(n-1) - (n-1)^2 = 4(n-1)$$

Hitchin: bijective correspondence between harmonic maps $T^2 \to S^3$ and hyperelliptic curves $\mathcal{C}$: $\eta^2 = f(\lambda)$ such that

- $f(\lambda)$ is real with respect to the real structure $\lambda \mapsto \bar{\lambda}^{-1}$.
- $f(\lambda)$ has no real zeros (i.e. no zeros on the unit circle).
- $f(\lambda)$ has at most simple zeros at $\lambda = 0$ and $\lambda = \infty$.
- Θ and Ψ are meromorphic differentials on $\mathcal{C}$ whose only singularities are double poles at $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ and which have no residues. Their principal parts are linearly independent over $\mathbb{R}$, and they satisfy
$$\sigma^*\Theta = -\Theta,\ \sigma^*\Psi = -\Psi,\ \rho^*\Theta = \bar{\Theta},\ \rho^*\Psi = \bar{\Psi}$$
where σ is the hyperelliptic involution $(\lambda, \eta) \mapsto (\lambda, -\eta)$ and ρ is the real structure induced from $\lambda \mapsto \bar{\lambda}^{-1}$.
- *The periods of Θ and Ψ are all integers.*
- $E(0)$ is a line bundle of degree $g + 1$ on $\mathcal{C}$, quaternionic with respect to the real structure $\sigma\rho$.

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\sum_i \partial_a X_i \partial^a X_i - \partial_a X_0 \partial^a X_0 \right], \qquad \sum_i X_i^2 = 1.$$

$$j = -g^{-1} dg \qquad g := \begin{pmatrix} X_4 + iX_3 & X_2 + iX_1 \\ -X_2 + iX_1 & X_4 - iX_3 \end{pmatrix} \in SU(2)$$

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \left[\frac{1}{2} \mathrm{tr}(j \wedge *j) + dX_4 \wedge *dX_4 \right]$$

$$dj - j \wedge j = 0, \quad d * j = 0,$$

Virasoro constraints (gauge $X_4 = \kappa\tau$) $\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa^2$

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\sum_i \partial_a X_i \partial^a X_i - \partial_a X_0 \partial^a X_0 \right], \qquad \sum_i X_i^2 = 1.$$

$$j = -g^{-1} dg \qquad g := \begin{pmatrix} X_4 + iX_3 & X_2 + iX_1 \\ -X_2 + iX_1 & X_4 - iX_3 \end{pmatrix} \in SU(2)$$

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \left[\frac{1}{2} \mathrm{tr}(j \wedge *j) + dX_4 \wedge *dX_4 \right]$$

$$dj - j \wedge j = 0, \quad d * j = 0,$$

Virasoro constraints (gauge $X_4 = \kappa\tau$) $\frac{1}{2}\mathrm{tr} j_\pm^2 = -\kappa^2$

$$J(x) = \frac{j - x * j}{1 - x^2}, \qquad F_J := dJ - J \wedge J = 0$$

$$\Omega(x, \sigma, \tau) = P\overleftarrow{\exp} \int_{\gamma(\sigma,\tau)} J(x, \sigma, \tau), \qquad [d - J, \Omega] = 0$$

- $[d - J, \Omega] = 0, \qquad \mathcal{C}:\ 0 = \det(y1_2 - \Omega(x))$
  $T := T(x, \sigma, \tau) = \mathrm{Tr}\,\Omega(x, \sigma, \tau)$
  $$y^2 - T\,y + 1 = 0$$
  a hyperelliptic curve, branched over $T^2 = 4$.
- $u(x, \sigma, \tau)\Omega(x, \sigma, \tau)u(x, \sigma, \tau)^{-1} = \mathrm{diag}\left(e^{ip(x)}, e^{-ip(x)}\right).$

- $[d - J, \Omega] = 0, \qquad \mathcal{C}:\ 0 = \det(y 1_2 - \Omega(x))$
  $T := T(x, \sigma, \tau) = \mathrm{Tr}\, \Omega(x, \sigma, \tau)$
  $$y^2 - T\, y + 1 = 0$$
  a hyperelliptic curve, branched over $T^2 = 4$.
- $u(x, \sigma, \tau)\Omega(x, \sigma, \tau)u(x, \sigma, \tau)^{-1} = \mathrm{diag}\left(e^{ip(x)}, e^{-ip(x)}\right)$.
- $g \in SU(2) \Rightarrow j^\dagger = -j \Rightarrow \Omega(x, \tau, \sigma) = \Omega(\bar{x}, \tau, \sigma)^{-1}$
  $\mathcal{C}$ real structure $\Rightarrow$ BP's in complex conjugate pairs or real.
- Flows $J(x)$ poles at $x = \pm 1$: $0 = \oint_{\mathfrak{a}_i} dp,\ \dfrac{1}{2\pi}\oint_{\mathfrak{b}_i} dp = n_i \in \Lambda$

$$dp(x^\pm) = \begin{cases} \mp d\left(\frac{\pi\kappa}{x-1}\right) + O\left((x-1)^0\right) & \text{as } x \to +1, \\ \mp d\left(\frac{\pi\kappa}{x+1}\right) + O\left((x+1)^0\right) & \text{as } x \to -1. \end{cases}$$

$$dp =_{x\to\infty} \frac{2\pi q_R}{\sqrt{\lambda}}\frac{dx}{x^2} + O\left(\frac{1}{x^3}\right), \qquad dp =_{x\to 0} \frac{2\pi q_L}{\sqrt{\lambda}}\, dx + O\left(x\right)$$