The Geometry of Monopoles: New and Old III

H.W. Braden

Varna, June 2011

Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Recall: To construct a $su(2)$ charge n monopole we need

- Curve $\mathcal{C} \subset T\mathbb{P}^1 : 0 = P(\eta, \zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta)$

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- $\deg a_r(\zeta) \leq 2r$ $\quad a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$

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- Ercolani-Sinha Constraints:
  1. $\mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}.$
  2. $\Omega = \dfrac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\, d\zeta, \quad \oint_{\mathfrak{es}} \Omega = -2\beta_0$

  $\mathfrak{es} = \mathbf{n}\cdot\mathfrak{a} + \mathbf{m}\cdot\mathfrak{b}$ impose g *transcendental constraints on* $\mathcal{C}$

$$\sum_{j=2}^n (2j+1) - g = (n+3)(n-1) - (n-1)^2 = 4(n-1)$$

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- Flows and Theta Divisor: $s\mathbf{U} + \mathbf{C} \notin \Theta$
- Symmetry aids calculation of τ, $\mathbf{U}$, K_Q

Possible Charge 3 Curve

A trigonal curve and its homology

- $w^3 = \prod_{i=1}^{6}(z - \lambda_i) \qquad \mathcal{R} : (z, w) \to (z, \rho w), \ \rho = \exp\{2i\pi/3\}$

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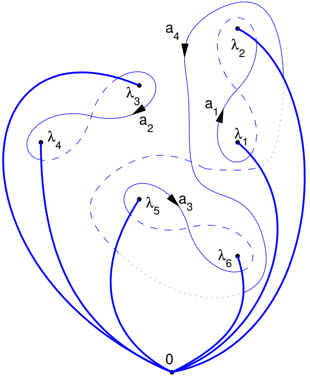
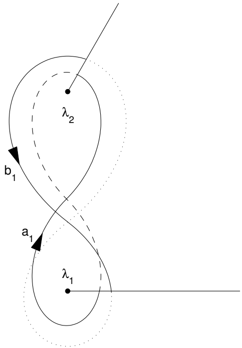
- $\mathcal{R}(\mathfrak{b}_i) = \mathfrak{a}_\mathfrak{i}, \quad i = 1, 2, 3, \quad \mathcal{R}(\mathfrak{b}_4) = -\mathfrak{a}_4 \qquad \mathrm{Aut}(\mathcal{C}) = \mathrm{C}_3$





Possible Charge 3 Curve

Differentials and periods (Wellstein, 1899; Matsumoto, 2000; BE 2006)

$$\mathrm{d}u_1 = \frac{\mathrm{d}z}{w}, \quad \mathrm{d}u_2 = \frac{\mathrm{d}z}{w^2}, \quad \mathrm{d}u_3 = \frac{z\mathrm{d}z}{w^2}, \quad \mathrm{d}u_4 = \frac{z^2\mathrm{d}z}{w^2}$$

$$\mathcal{A} = (\mathcal{A}_{ki}) = \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,4} = (\mathbf{x}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

$$\mathcal{B} = H\mathcal{A}\Lambda, \qquad H = \mathrm{diag}(1,1,1,-1),\ \Lambda = \mathrm{diag}(\rho, \rho^2, \rho^2, \rho^2)$$

$$\sum_i \left(\oint_{\mathfrak{a}_i} \mathrm{d}u_k \oint_{\mathfrak{b}_i} \mathrm{d}u_l - \oint_{\mathfrak{b}_i} \mathrm{d}u_k \oint_{\mathfrak{a}_i} \mathrm{d}u_l \right) = 0 \Leftrightarrow 0 = \mathbf{x}^T H\mathbf{b} = \mathbf{x}^T H\mathbf{c} = \mathbf{x}^T H\mathbf{d}$$

$$\tau_{\mathfrak{b}} = \mathcal{A}\mathcal{B}^{-1} = \rho\left(H - (1-\rho)\frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T H\mathbf{x}} \right)$$

$\mathrm{Im}\,\tau_{\mathfrak{b}}$ is positive definite if and only if $\bar{\mathbf{x}}^T H\mathbf{x} < 0$

Theorem

$$\mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m} \Longleftrightarrow$$

$$\mathbf{x} = \xi(H\mathbf{n} + \rho^2\mathbf{m}), \qquad \xi = \frac{6\chi^{\frac{1}{3}}}{[\mathbf{n}^T H\mathbf{n} - \mathbf{m}.\mathbf{n} + \mathbf{m}^T H\mathbf{m}]}$$

where ξ is real

$$\frac{x_1}{n_1 + \rho^2 m_1} = \frac{x_2}{n_2 + \rho^2 m_2} = \frac{x_3}{n_3 + \rho^2 m_3} = \frac{x_4}{-n_4 + \rho^2 m_4} = \xi,$$

$$x_i/x_j \in \mathbb{Q}[\rho] \qquad \bar{\mathbf{x}}^T H\mathbf{x} < 0 \Longleftrightarrow \frac{\bar{\mathbf{x}}^T H\mathbf{x}}{|\xi|^2} = [\mathbf{n}^T H\mathbf{n} - \mathbf{m}.\mathbf{n} + \mathbf{m}^T H\mathbf{m}]$$

$$= \sum_{i=1}^{3}(n_i^2 - n_i m_i + m_i^2) - n_4^2 - m_4^2 - m_4 n_4 < 0.$$

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- $\mathrm{Aut}(\mathcal{C}) \to \mathsf{C}_3 \times S_3\ (\zeta,\eta) \mapsto (\rho\zeta,\eta),\ (\zeta,\eta) \mapsto (-1/\zeta, -\eta/\zeta^2)$

$$\mathcal{I}_1(\alpha) = \int_0^\alpha \frac{\mathrm{d}z}{w} = -\frac{2\pi\sqrt{3}\alpha}{9}\,{}_2F_1\left(\frac{1}{3},\frac{1}{3};1;-\alpha^6\right)$$

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- $$\begin{aligned} x_1 &= -(2\mathcal{J}_1 + \mathcal{I}_1)\rho - 2\mathcal{I}_1 - \mathcal{J}_1, & x_2 &= \rho\, x_1, \\ x_3 &= \rho^2 x_1, & x_4 &= 3(\mathcal{J}_1 - \mathcal{I}_1)\rho + 3\mathcal{J}_1, \end{aligned}$$

To each pair of relatively prime integers $(n, m) = 1$ for which

$$(m+n)(m-2n) < 0$$

we obtain a solution to the Ercolani-Sinha constraints for our curve with $\mathbf{n} = (n, m-n, -m, 2n-m)$, $\mathbf{m} = (m, -n, n-m, -3n)$ as follows. First we solve for t, where

$$\frac{2n-m}{m+n} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, 1-t)}.$$

Then

$$b = \frac{1-2t}{\sqrt{t(1-t)}}, \qquad t = \frac{-b+\sqrt{b^2+4}}{2\sqrt{b^2+4}},$$

and with $\alpha^6 = t/(1-t)$ we obtain χ from

$$\chi^{\frac{1}{3}} = -(n_1+m_1)\,\frac{2\pi}{3\sqrt{3}}\,\frac{\alpha}{(1+\alpha^6)^{\frac{1}{3}}}\;{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t).$$

Ramanujan (1914)

$$\frac{4}{\pi} = \sum_{m=0}^{\infty} \frac{(1+6m)(\frac{1}{2})_m(\frac{1}{2})_m(\frac{1}{2})_m}{(m!)^3 4^m}; \quad \frac{27}{4\pi} = \sum_{m=0}^{\infty} \frac{(2+15m)(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3 \left(\frac{27}{2}\right)^m}$$

$$\frac{15\sqrt{3}}{2\pi} = \sum_{m=0}^{\infty} \frac{(44+33m)(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3 \left(\frac{125}{4}\right)^m}$$

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$$\tau=i\frac{K'}{K},\quad K(k)=\frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2),\quad K'(k)=K(k'),\ \ k'^2=1-k^2$$

If $k_1=\dfrac{1-k'}{1+k'}$ then $\tau_1=2\tau$. Modular equation of degree n:

$$n\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}$$

$$(\alpha\beta)^{1/4}+((1-\alpha)(1-\beta))^{1/4}=1\implies n=3$$

Modular equation of degree n and signature r $(r = 2, 3, 4, 6)$

$$n\,\frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}$$

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$$n = 2,\ r = 3 \Longrightarrow (\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} = 1$$

$$\alpha = 1/2 \Longrightarrow \beta^{1/3} + (1-\beta)^{1/3} = 2^{1/3} \Longrightarrow \beta = \frac{1}{2} + \frac{5\sqrt{3}}{18}$$

$n{=}5,\ r{=}3 \Rightarrow (\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} + 3(\alpha\beta(1-\alpha)(1-\beta))^{1/6} = 1$

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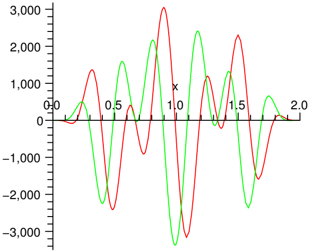
n	m	$\frac{2n-m}{m+n}$	t	b
2	1	1	$\frac{1}{2}$	0
1	0	2	$\frac{1}{2}+\frac{5\sqrt{3}}{18}$	$-5\sqrt{2}$
1	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{5\sqrt{3}}{18}$	$5\sqrt{2}$
4	-1	3	$(63+171\sqrt[3]{2}-18\sqrt[3]{4})/250$	$\frac{1}{3}(44+38\sqrt[3]{2}+26\sqrt[3]{4})$
5	-2	4	$\frac{1}{2}+\frac{153\sqrt{3}-99\sqrt{2}}{250}$	$9\sqrt{458+187\sqrt{6}}$

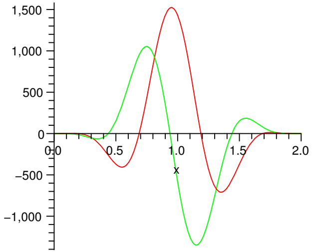
Symmetric 3-monopoles

The H3 constraint

Flows and Theta Divisor: $\theta(s\mathbf{U}+\mathbf{C}) \neq 0$

$$\mathbf{C} = K_Q + \phi_Q\left((n-2)\sum_{k=1}^{n} \infty_k\right), \quad Q \text{ a branchpoint } \mathbf{C} = K_Q$$





Conjecture: No solutions to H3 apart from $(\mathbf{n},\mathbf{m}) \in \{(1,0),(1,1)\}$

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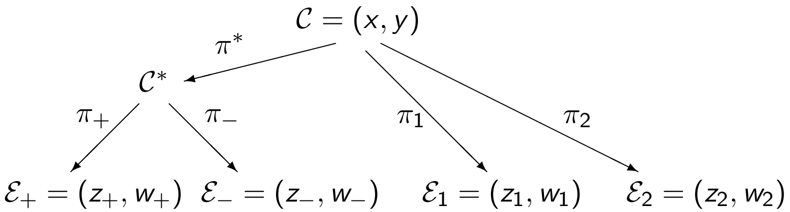
The H3 constraint and Symmetry



$$\mathcal{C} = (x, y)$$

$$\mathcal{C} \xrightarrow{\pi^*} \mathcal{C}^*, \quad \mathcal{C} \xrightarrow{\pi_1} \mathcal{E}_1, \quad \mathcal{C} \xrightarrow{\pi_2} \mathcal{E}_2, \quad \mathcal{C}^* \xrightarrow{\pi_+} \mathcal{E}_+, \quad \mathcal{C}^* \xrightarrow{\pi_-} \mathcal{E}_-$$

$$\mathcal{E}_+ = (z_+, w_+) \quad \mathcal{E}_- = (z_-, w_-) \quad \mathcal{E}_1 = (z_1, w_1) \quad \mathcal{E}_2 = (z_2, w_2)$$

- $\mathcal{C}$ covers the $g_{\mathcal{C}^*} = 2$ hyperelliptic curve

$$\mathcal{C}^* = \{(\mu, \nu) | \nu^2 = (\mu^3 + b)^2 + 4\}$$

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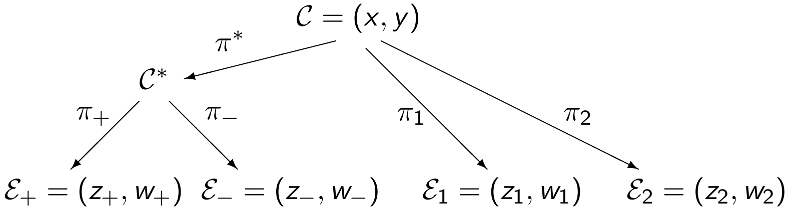

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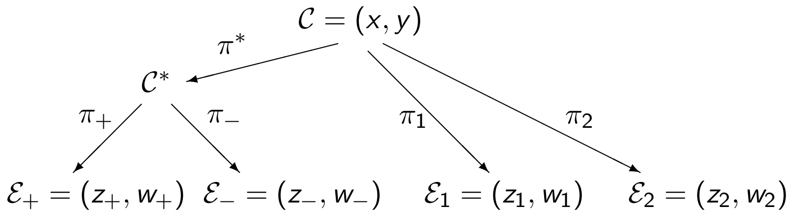

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- Parameterize $\dfrac{(2\imath - b)^{\frac{1}{3}}}{(b^2 + 4)^{\frac{1}{6}}} = \dfrac{1 + 2\rho + p}{1 + 2\rho - p}$ with $p = \dfrac{3\vartheta_3^2(0|3\tau)}{\vartheta_3^2(0|\tau)}$

$$k_+ = \frac{\vartheta_2^2(0|\tau)}{\vartheta_3^2(0|\tau)}, \quad k_- = \frac{\vartheta_2^2(0|3\tau)}{\vartheta_3^2(0|3\tau)}$$

The Symmetric Monopole

Rewriting H2

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b \in \mathbb{R}$$

satisfies **H1** and **H2** $\Leftrightarrow \exists\ n, m\ (n,m) = 1,\ (m+n)(m-2n) < 0$

$$b = b(m,n) = -\frac{\sqrt{3}(p(m,n)^6 - 45p(m,n)^4 + 135p(m,n)^2 - 27)}{9p(m,n)(p(m,n)^4 - 10p(m,n)^2 + 9)}$$

$$p(m,n) = \frac{3\vartheta_3^2\left(0|\frac{\mathcal{T}(m,n)}{2}\right)}{\vartheta_3^2\left(0|\frac{\mathcal{T}(m,n)}{6}\right)}, \qquad \mathcal{T}(m,n) = 2\imath\sqrt{3}\frac{n+m}{2n-m}$$

Expression for $\chi = \chi(m,n)$ can be given.

What about H3? Reduces to questions of θ for $\mathcal{C}^*$.

The Humbert Variety

τ the period matrix of a genus 2 curve $\mathcal{C}^*$.

- $\tau \in \mathcal{H}_\Delta$ if there exist $q_i \in \mathbb{Z}$
  $q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0$
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- Bierman-Humbert: $\tau \in \mathcal{H}_{h^2} \Rightarrow \exists\, \mathfrak{S} \in \mathrm{Sp}(4, \mathbb{Z})$, such that
  $\mathfrak{S} \circ \tau = \widetilde{\tau} = \begin{pmatrix} \widetilde{\tau}_{11} & \frac{1}{h} \\ \frac{1}{h} & \widetilde{\tau}_{22} \end{pmatrix}$

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  $$\mathfrak{S} \circ \tau = \widetilde{\tau} = \begin{pmatrix} \widetilde{\tau}_{11} & \frac{1}{h} \\ \frac{1}{h} & \widetilde{\tau}_{22} \end{pmatrix}$$
- $$\theta(z_1, z_2 \,|\, \widetilde{\tau}) = \sum_{k=0}^{h-1} \vartheta_3\left(z_1 + \frac{k}{h} \,\Big|\, \widetilde{\tau}_{1,1}\right) \theta\begin{bmatrix} \frac{k}{h} \\ 0 \end{bmatrix}\left(hz_2 \,|\, h^2\widetilde{\tau}_{2,2}\right)$$

The Humbert Variety

τ the period matrix of a genus 2 curve $\mathcal{C}^*$.

- $\tau \in \mathcal{H}_\Delta$ if there exist $q_i \in \mathbb{Z}$
  $q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0$
  $q_3^2 - 4(q_1 q_5 + q_2 q_4) = \Delta$
- $\mathcal{C}^*$ covers elliptic curves $\mathcal{E}_\pm \Leftrightarrow \Delta = h^2 \geq 1,\ h \in \mathbb{N}$.
- Bierman-Humbert: $\tau \in \mathcal{H}_{h^2} \Rightarrow \exists\, \mathfrak{S} \in \mathrm{Sp}(4, \mathbb{Z})$, such that
  $\mathfrak{S} \circ \tau = \widetilde{\tau} = \begin{pmatrix} \widetilde{\tau}_{11} & \frac{1}{h} \\ \frac{1}{h} & \widetilde{\tau}_{22} \end{pmatrix}$
- $\theta(z_1, z_2 \,|\, \widetilde{\tau}) = \sum_{k=0}^{h-1} \vartheta_3\left(z_1 + \frac{k}{h} \,|\, \widetilde{\tau}_{1,1}\right) \theta\begin{bmatrix} \frac{k}{h} \\ 0 \end{bmatrix}\left(hz_2 \,|\, h^2\widetilde{\tau}_{2,2}\right)$
- $\theta(z_1, z_2 \,|\, \widetilde{\tau}) = \vartheta_3\left(z_1 \,|\, \widetilde{\tau}_{11}\right)\vartheta_3\left(2z_2 \,|\, 4\widetilde{\tau}_{22}\right) + \vartheta_3\left(z_1 + 1/2 \,|\, \widetilde{\tau}_{11}\right)\vartheta_2\left(2z_2 \,|\, 4\widetilde{\tau}_{22}\right)$

- Bierman-Humbert+Weierstrass-Poincáre+Martens $\theta(\lambda\mathbf{U} - \widetilde{\mathbf{K}};\, \tau_{\text{monopole}}) = 0$ for $\lambda \in [0, 2] \Leftrightarrow$ at least one of the functions ($k = -1, 0, 1 \mod 3$)

$$h_k(y) := \frac{\vartheta_3}{\vartheta_2}\left(i\sqrt{3}\,y + \frac{k\,\mathcal{T}}{3}\,\Big|\,\mathcal{T}\right) + (-1)^k\,\frac{\vartheta_2}{\vartheta_3}\left(y + \frac{k}{3}\,|\,\frac{\mathcal{T}}{3}\right)$$

also vanishes. $y := y(\lambda) = \lambda\,(n+m)\rho/3, \quad \mathcal{T} = 2i\sqrt{3}\dfrac{n+m}{2n-m}$

$\rho = \exp(2\pi\imath/3)$

The Symmetric Monopole

H3 and an elliptic function conjecture

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- Numerically $h_k(y) = 0 \Leftrightarrow h_k(\rho y) = 0 \Leftrightarrow h_k(\rho^2 y) = 0$ True?

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- When does $s\mathbf{U} + \mathbf{C} \notin \Theta$?