The Geometry of Monopoles: New and Old IV

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Varna, June 2011

Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Recall

- The only spectral curves of a BPS monopole of the form $\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$ have $b = \pm 5\sqrt{2}$, $\chi^{\frac{1}{3}} = -\frac{1}{6}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{2^{\frac{1}{6}}\,\pi^{\frac{1}{2}}}$. These correspond to tetrahedrally symmetric monopoles.

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Invariant curves yield symmetric monopoles.

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- Consider cyclic space-time symmetry

Cyclically Symmetric Monopoles

Spectral Curves

- $\omega = \exp(2\pi i/n)$, $p = \omega^{1/2}$ $q = 0$ $\quad (\eta, \zeta) \to (\omega\eta, \omega\zeta)$
  $\eta^i \zeta^j$ invariant for $i + j \equiv 0 \mod n$

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- Affine Toda Spectral Curve $\; y = \nu - \frac{(-1)^n |\beta|^2}{\nu}$

$$y^2 = (x^n + a_2 x^{n-2} + \ldots + a_n)^2 - 4(-1)^n |\beta|^2$$

- $\mathcal{C}_{\text{monopole}}$ is an unbranched $n:1$ cover $\mathcal{C}_{\text{Toda}}$
  $g_{\text{monopole}} = (n-1)^2$, $g_{\text{Toda}} = (n-1)$

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- Implementation in terms of curves, period matrices, theta functions etc.

Cyclic Nahm eqns. $\equiv$ Affine Toda eqns.

Sutcliffe Ansatz

$$
\begin{aligned}
T_1 + iT_2 &= (T_1 - iT_2)^T \\
&= \begin{pmatrix} 0 & e^{(q_1-q_2)/2} & 0 & \dots & 0 \\ 0 & 0 & e^{(q_2-q_3)/2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} \\ e^{(q_n-q_1)/2} & 0 & 0 & \dots & 0 \end{pmatrix} \\
T_3 &= -\frac{i}{2}\,\mathrm{Diag}(p_1, p_2, \dots, p_n)
\end{aligned}
$$

$$
\frac{d}{ds}(T_1 + iT_2) = i[T_3, T_1 + iT_2] \Rightarrow p_i - p_{i+1} = \dot{q}_i - \dot{q}_{i+1}
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Sutcliffe Ansatz C'td

- p_i, q_i real

$$H = \frac{1}{2}\left(p_1^2 + \ldots + p_n^2\right) - \left[e^{q_1-q_2} + e^{q_2-q_3} + \ldots + e^{q_n-q_1}\right].$$

Toda $\Rightarrow$ Nahm   Affine Toda eqns. $\subset$ Cyclic Nahm eqns.

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- $G \subset SO(3)$ acts on triples $\mathbf{t} = (T_1, T_2, T_3) \in \mathbb{R}^3 \otimes SL(n, \mathbb{C})$ via natural action on $\mathbb{R}^3$ and conjugation on $SL(n, \mathbb{C})$
- $g' \in SO(3)$ and $g = \rho(g') \in SL(n, \mathbb{C})$. Invariance of curve $\Rightarrow$

$$\begin{aligned} g(T_1 + iT_2)g^{-1} &= \omega(T_1 + iT_2), \\ gT_3g^{-1} &= T_3, \\ g(T_1 - iT_2)g^{-1} &= \omega^{-1}(T_1 - iT_2). \end{aligned}$$

Cyclically Symmetric Monopoles

Cyclic Nahm eqns. $\equiv$ Affine Toda eqns.

- $SO(3)$ action on $SL(n,\mathbb{C}) \sim \underline{2n-1} \oplus \underline{2n-3} \oplus \ldots \oplus \underline{5} \oplus \underline{3}$

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- Kostant $\Rightarrow \rho(SO(3))$ principal three dimensional subgroup.
- $g = \rho(g') = \exp\left[\frac{2\pi}{n} H\right]$, H semi-simple, generator Cartan TDS
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$$T_1 + iT_2 = \sum_{\alpha\in\hat{\Delta}} e^{(\alpha,\tilde{q})/2}\, E_\alpha, \qquad T_3 = -\frac{i}{2}\sum_j \tilde{p}_j\, H_j$$

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  $\tilde{q}_i \in \mathbb{R}$ with $SU(n)$ conjug. $+$ overall $SO(3)$ rotation.
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- Theorem *Any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations.*

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$$\theta[\mathbf{C}](\pi^* z;\, \widehat{\tau}_{\mathrm{monopole}}) = c \prod_{i=1}^{n} \theta[\mathbf{e}_i](z;\, \tau_{\mathrm{Toda}})$$

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- $$\frac{\theta(3z_1, z_2, z_2, z_2; \widehat{\tau})}{\theta(z_1, z_2; \tau)\theta(z_1 + 1/3, z_2; \tau)\theta(z_1 - 1/3, z_2; \tau)} = c$$

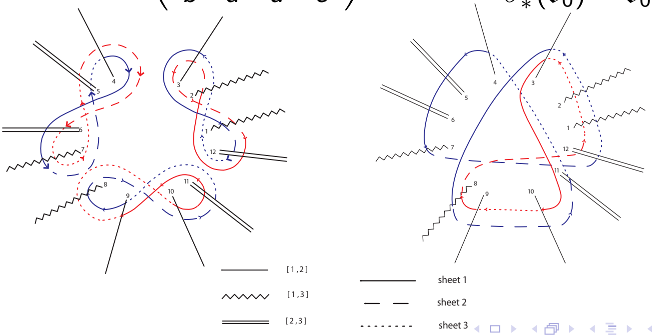
The C_3 cyclically symmetric spectral curve of genus 4

$$\hat{\mathcal{C}}: \quad w^3 + \alpha w z^2 + \beta z^6 + \gamma z^3 - \beta = 0 \ (\alpha = 0 \text{ Symmetric Mon}^{\text{ple}})$$

$$C_3 : (z, w) \mapsto (\rho z, \rho w), \ \rho = \exp(2\pi i/3)$$

$$\tau_{\hat{\mathcal{C}}_{\text{monopole}}} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\sigma_*^k(\mathfrak{a}_i) = \mathfrak{a}_{i+k}$$
$$\sigma_*^k(\mathfrak{b}_i) = \mathfrak{b}_{i+k}$$
$$\sigma_*^k(\mathfrak{a}_0) = \mathfrak{a}_0$$
$$\sigma_*^k(\mathfrak{b}_0) \sim \mathfrak{b}_0$$



The $\mathbb{C}_3$ Cyclically Symmetric Monopole

The spectral curve of genus 2

$$\mathcal{C} = \hat{\mathcal{C}}/\mathbb{C}_3 : \qquad y^2 = (x^3 + \alpha x + \gamma)^2 + 4\beta^2$$

$$\tau = \begin{pmatrix} \frac{a}{3} & b \\ b & c + 2d \end{pmatrix}$$

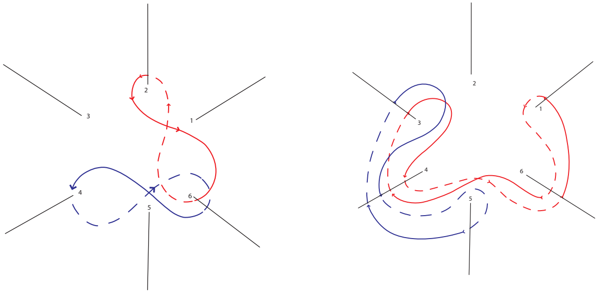


Figure: Projection of the previous basis

$\mathcal{C}_3$ Cyclically Symmetric Monopoles
ES conditions

- $\mathfrak{c} := \pi(\mathfrak{es})$
  $Y^2 = (X^3 + a\,X + g)^2 + 4$

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C_3 Cyclically Symmetric Monopoles
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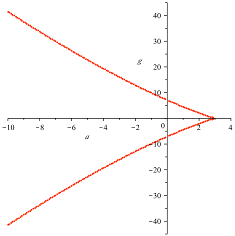

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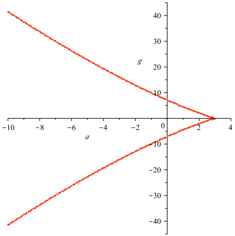


- With $a = \alpha/\beta^{2/3}$, $g = \gamma/\beta$ and β defined by

$$6\beta^{1/3} = \oint_{\mathfrak{c}} \frac{X\mathrm{d}X}{Y}$$

we may recover the monopole spectral curve.

C_3 Cyclically Symmetric Monopoles
Properties

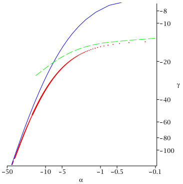
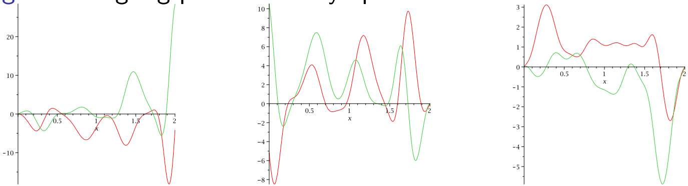


Figure: A log-log plot of the asymptotic behaviour of α versus γ



C_3 Cyclically Symmetric Monopoles
The Richelot Transform

- $\oint_{\mathfrak{c}} \frac{\mathrm{d}X}{Y}$ $\qquad Y^2 = (X^3 + a\,X + g)^2 + 4$

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- $a, b \in \mathbb{R}_+ \quad a_0 = a,\ b_0 = b,\ a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}$

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$\mathcal{E}_n \to \mathcal{E}_{n+1}, \qquad \mathcal{E}_n : \quad y_n^2 = x_n(x_n - a_n^2)(x_n - b_n^2)$

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-

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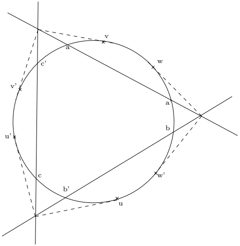

FIGURE 1. Roots of P, Q, R and U, V, W

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$$V(x) = (x-v)(x-v')$$
$$W(x) = (x-w)(x-w')$$

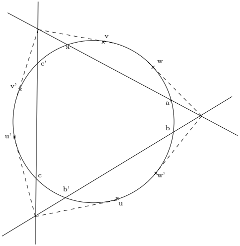
- Degree 2 correspondence $\mathcal{Z} \subset \mathcal{C} \times \mathcal{C}'$
  $\mathcal{C}:\ y^2 + P(x)Q(x)R(x) = 0,\ \mathcal{C}':\ \Delta y'^2 + U(x')V(x')W(x') = 0$

$$\mathcal{Z}: \quad \begin{cases} P(x)U(x') + Q(x)V(x') = 0, \\ yy' = P(x)U(x')(x-x'). \end{cases}$$

C_3 Cyclically Symmetric Monopoles

The Richelot Transform

- $\oint_{\mathfrak{c}} \frac{\mathrm{d}X}{Y}$ $\qquad Y^2 = (X^3 + a\,X + g)^2 + 4$
-



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- $$\int_a^{a'} \frac{dx}{\sqrt{-P(x)Q(x)R(x)}} = 2\sqrt{\Delta}\int_v^w \frac{dx}{\sqrt{-U(x)V(x)W(x)}}$$

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- Flows and Theta Divisor:
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