Krichever formal groups and the deformed Baker-Akhiezer function.

Elena Yu. Bunkova

⟨bunkova@mi.ras.ru⟩

Steklov Mathematical Institute, Russian Academy of sciences,

Laboratory of geometric methods in mathematical physics, MSU

Moscow

in collaboration with V.M.Buchstaber

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The formal group.

A commutative one-dim formal group law (or shortly formal group) over a commutative associative ring A is the formal series

$$F(u,v) = u + v + \sum a_{i,j} u^i v^j, \quad a_{i,j} \in A, \quad i > 0,\ j > 0,$$

satisfying the conditions

$$F(u,v) = F(v,u), \qquad F\Big(u, F(v,w)\Big) = F\Big(F(u,v), w\Big).$$

Let $F_a(u,v) = u + v$. For any formal group $F(u,v) \in A[[u,v]]$ there exists an isomorphism $f : F_a \to F$ over $A \otimes \mathbb{Q}$.
The series $f(t) \in A \otimes \mathbb{Q}[[t]]$ uniquely defined by the conditions

$$f(t_1 + t_2) = F\Big(f(t_1), f(t_2)\Big), \quad f(0) = 0, \quad f'(0) = 1$$

is the *exponential* of the formal group $F(u,v)$.

Examples of formal groups and their exponentials.

$$F(u,v) = u + v - \mu_1 uv, \qquad f(t) = \frac{1}{\mu_1}\Big(1 - \exp(-\mu_1 t)\Big).$$

$$F(u,v) = \frac{u+v}{1+\mu_2 uv}, \qquad f(t) = \frac{1}{\sqrt{\mu_2}}\, th\Big(\sqrt{\mu_2}\, t\Big).$$

$$F(u,v) = \frac{u\sqrt{1-2\delta v^2+\varepsilon v^4} + v\sqrt{1-2\delta u^2+\varepsilon u^4}}{1-\varepsilon u^2 v^2}, \qquad f(t) = sn(t),$$

where $sn(t)$ is the Jacobi elliptic sine:

$$(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4.$$

The general Weierstrass model of the elliptic curve

$$Y^2Z + \mu_1 XYZ + \mu_3 YZ^2 = X^3 + \mu_2 X^2 Z + \mu_4 XZ^2 + \mu_6 Z^3$$

depends on 5 parameters $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_6)$.

The geometric group structure „$+$“ on the elliptic curve:
The points P, Q, R of the curve are on a straight line
if and only if $P + Q + R = 0$.

Let $O = (0 : 1 : 0)$ be the zero of the geometric group structure.

For $P + Q + R = 0$ and $R + \overline{R} + O = 0$ we get $P + Q = \overline{R}$.

In the chart $Z \neq 0$ in *Weierstrass coordinates* $x = X/Z$ and $y = Y/Z$ the curve takes the form

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

In the chart $Y \neq 0$ in *Tate coordinates* $u = -X/Y$ and $s = -Z/Y$ the curve takes the form

$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3.$$

In the chart $X \neq 0$ with coordinates $v = Y/X$ and $w = Z/X$ the curve takes the form

$$vw(v + \mu_1 + \mu_3 w) = 1 + \mu_2 w + \mu_4 w^2 + \mu_6 w^3.$$

The gradings are
$\deg X = -4$, $\deg Y = -6$, $\deg Z = 0$, $\deg \mu_i = -2i$.

The Tate coordinates of the elliptic curve.

In the chart $\{Y \neq 0\}$ with $u = -X/Y$ and $s = -Z/Y$ the curve takes the form

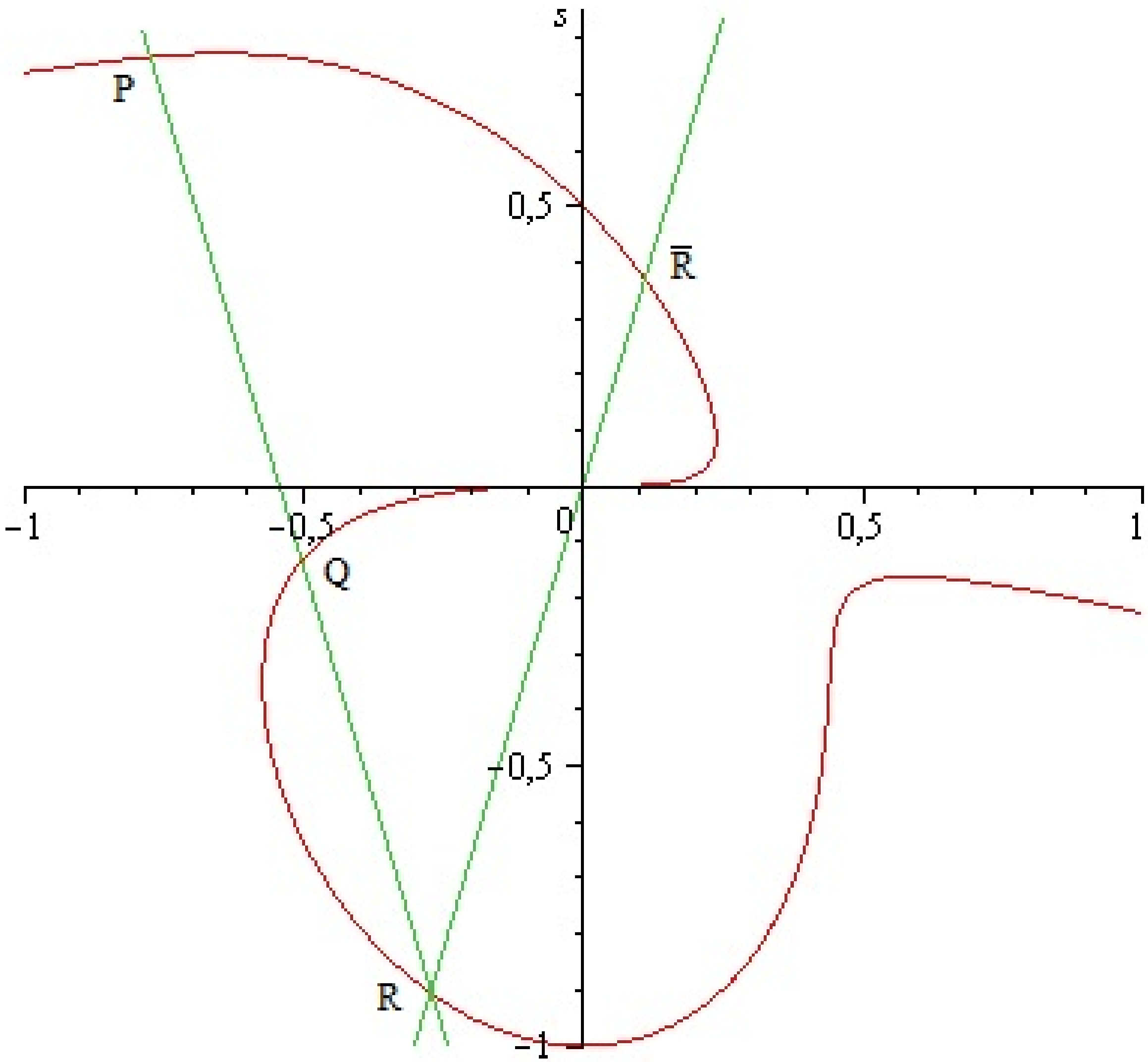
$$s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3.$$

We get the formal series $s(u) \in \mathbb{Z}[\mu][[u]]$:

$$s = u^3 + \mu_1 u^4 + (\mu_1^2 + \mu_2)u^5 + (\mu_1^3 + 2\mu_1\mu_2 + \mu_3)u^6 + \\ + (\mu_1^4 + 3\mu_1^2\mu_2 + \mu_2^2 + 3\mu_1\mu_3 + \mu_4)u^7 + \dots$$

The coordinates $(u, s(u))$ are the arithmetic Tate coordinates. They define the Tate unifomization of the elliptic curve.

$$s = u^3 + 2\,u\,s + 4\,u^2 s + s^2 + 2\,u\,s^2 + 2\,s^3$$



The elliptic formal group.

Let $P = \big(u, s(u)\big)$, $Q = \big(v, s(v)\big)$, $R = \big(w, s(w)\big)$ and $\overline{R} = \big(\bar{w}, s(\bar{w})\big)$. The geometric group structure on the elliptic curve defines the series $\mathcal{F}_{El}(u,v)$ over $\mathbb{Z}[\mu]$:

$$\mathcal{F}_{El}(u,v) = \bar{w}.$$

Theorem. (General elliptic formal group)

$$\mathcal{F}_{El}(u,v) = \left(u + v - uv\frac{(\mu_1 + \mu_3 m) + (\mu_4 + 2\mu_6 m)k}{(1 - \mu_3 k - \mu_6 k^2)}\right) \times$$
$$\times \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 k - \mu_6 k^2)},$$

where $\quad \big(u, s(u)\big) \in V_\mu \quad$ and $\quad m = \dfrac{s(u) - s(v)}{u - v},$

$$k = \frac{us(v) - vs(u)}{u - v}, \qquad n = m + uv\frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 - \mu_3 k - \mu_6 k^2)}.$$

Formal groups for curves with automorphisms. The map $u \to \alpha u$ defines an automorphism of an elliptic formal group.

$\alpha = 2$, $\mu = (0, 0, \mu_2, \mu_4, \mu_6)$. Then

$$F(u,v) = u + v + b\frac{(\mu_2 + 2\mu_4 m + 3\mu_6 m^2)}{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}.$$

$\alpha = 3$, $\mu = (0, 0, \mu_3, 0, \mu_6)$. Then

$$F(u,v) = \frac{(u+v)(1 + \mu_6 m^3) + \mu_3 m^2 + 3\mu_6 m^2 b}{(1 + \mu_6 m^3)(1 - \mu_3 b) - \frac{bm}{uv}\mu_3(1 - \mu_3 b - \mu_6 b^2)}.$$

$\alpha = 4$, $\mu = (0, 0, 0, \mu_4, 0) \neq \mathbf{0}$.

$$f^2(f'^2 + 4\mu_4 f^4 - 1) = 0,$$

$$F(u,v) = u + v + \frac{2\mu_4 m b}{1 + \mu_4 m^2}.$$

$\alpha = 6$, $\mu = (0, 0, 0, 0, \mu_6) \neq \mathbf{0}$

$$f'^3 + 3f'^2 + 27\mu_6 f^6 - 4 = 0,$$

$$F(u,v) = u + v + \frac{3\mu_6 m^2 b}{1 + \mu_6 m^3}.$$

An elliptic function is a meromorphic function on the torus $\mathbb{T} = \mathbb{C}/\Gamma$, where $\Gamma \in \mathbb{C}$ is a grid of rank 2.

$$f(t+2\omega_1) = f(t), \quad f(t+2\omega_2) = f(t), \quad \text{where} \quad \mathbf{Im}(\frac{\omega_1}{\omega_2}) \neq 0.$$

Elliptic functions form a differential field, algebraicly generated by the functions $\wp(t)$ and $\wp'(t)$.
The Weierstrass function $\wp(t)$ is the unique even elliptic function on $\mathbb{C}$ with periods $2\omega_1$, $2\omega_2$ and double poles at grid points such that

$$\lim_{t\to 0}\left(\wp(t) - \frac{1}{t^2}\right) = 0.$$

We will need the Weierstrass σ-function, which is an entire function on the universal covering of $\mathbb{T}$ and is defined by the condition $(\ln\sigma)'' = -\wp$ and the initial conditions $\sigma(0) = 0$, $\sigma'(0) = 1$.

Theorem. The exponential of the general elliptic formal group is

$$f_{El}(t) = -2\frac{\wp(t) - \frac{1}{12}(\mu_1^2 + 4\mu_2)}{\wp'(t) - \mu_1(\wp(t) - \frac{1}{12}(\mu_1^2 + 4\mu_2)) - \mu_3}$$

where $\wp(t) = \wp(t; g_2(\mu), g_3(\mu))$.

$f_{El}(t)$ is an elliptic function of order 3 for $\mu_6 \neq 0$
and of order 2 for $\mu_6 = 0$ (in the case of a non-degenerate curve).

Example of a degenerate curve $\mu = (\mu_1, \mu_2, 0, 0, 0)$.

$$\wp(t) = \frac{(a-b)^2}{4}\left(\left(\frac{e^{at} + e^{bt}}{e^{at} - e^{bt}}\right)^2 - \frac{2}{3}\right) = \frac{1}{t^2} + \frac{(a-b)^4}{240}t^2 + \ldots$$

where $\mu_1 = a + b$, $\mu_2 = -ab$. The formal group is rational

$$F(u,v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}, \qquad \text{where} \qquad f(t) = \frac{e^{at} - e^{bt}}{ae^{at} - be^{bt}}.$$

We can rewrite the formula for the exponential as

$$\frac{1}{f_{El}(t)} = \frac{\mu_1}{2} - \frac{1}{2}\frac{\wp'(t) + \wp'(w)}{\wp(t) - \wp(v)},$$

where $\wp(t) = \wp(t; g_2(\mu), g_3(\mu))$, and v and w are determined by $\wp'(w) = -\mu_3$, $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$.

In the case $v = \pm w$ the compatibility condition for the last system is $\mu_6 = 0$.

For $\mu_6 = 0$ the solution of

$$(\ln \Phi(t))' = \frac{\mu_1}{2} - \frac{1}{f_{El}(t)} \tag{1}$$

is the well-known Baker-Akhiezer function of the elliptic curve.

The solution of eq. (1) in the general case is called the deformed Baker-Akhiezer function.

The Baker-Akhiezer function

$$\Phi(t) = \Phi(t, \tau) = \frac{\sigma(\tau + t)}{\sigma(t)\sigma(\tau)} e^{-\zeta(\tau)t}$$

gives a solution to the Lame equation

$$\left(\frac{d^2}{dt^2} - 2\wp(t)\right)\Phi(t) = \wp(\tau)\Phi(t).$$

In the vicinity of $t = 0$ we have

$$\Phi(t;\tau) = \frac{1}{t} - \frac{1}{2}\wp(\tau)t - \frac{1}{6}\wp'(\tau)t^2 + (t^3).$$

Let $2\omega_k$, $k = 1, 2$be the periods of the $\wp$-function. Then

$$\Phi(t + 2\omega_k;\tau) = \Phi(t;\tau)\,\mathrm{exp}(-2(\zeta(\tau)\omega_k - \eta_k\tau)),$$
$$\Phi(t + 2\omega_k;\omega_k) = \Phi(t;\omega_k), \quad \Phi(t;\tau + 2\omega_k) = \Phi(t;\tau).$$

Lemma.

The following operators annulate the Baker-Akhiezer function:

$$H_0 = 4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3} - t\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial \tau} - 1,$$

$$H_2 = 6g_3\frac{\partial}{\partial g_3} + \frac{1}{3}g_2^2\frac{\partial}{\partial g_3} - \zeta(t)\frac{\partial}{\partial t} - \zeta(\tau)\frac{\partial}{\partial \tau} - (\wp(t) + \wp(\tau) + \frac{1}{2}t\wp'(\tau)),$$

$$\mathcal{L}_1 = \frac{\partial}{\partial t} + P, \qquad \text{where} \qquad P = -\frac{1}{2}\frac{\wp'(t) - \wp'(\tau)}{\wp(t) - \wp(\tau)},$$

$$\mathcal{L}_2 = \frac{\partial^2}{\partial t^2} - 2\wp(t) - \wp(\tau),$$

$$\mathcal{L}_3 = 2\frac{\partial^3}{\partial t^3} - 6\wp(t)\frac{\partial}{\partial t} - 3\wp'(t) - \wp'(\tau).$$

Theorem.
The solution of the algebraic dynamical system on $\mathbb{C}^4$ with coordinates (z_1, z_2, x, y):

$$\partial_t z_1 = z_2, \qquad \partial_\tau z_1 = \frac{1}{2}(z_1^2 + z_2 - 3x),$$
$$\partial_t z_2 = 2z_1^3 - 6z_1 x - 2y, \qquad \partial_\tau z_2 = z_1^3 - 3z_1 x - y + z_1 z_2,$$
$$\partial_t x = 0, \qquad \partial_\tau x = y,$$
$$\partial_t y = 0, \qquad \partial_\tau y = 6x^2 - \frac{1}{2}(z_2^2 - z_1^4 + 6xz_1^2 + 4yz_1 + 3x^2).$$

passing at $t = 0, \tau = 0$ through (z_1^0, z_2^0, x^0, y^0)
where $3x^0 \neq (z_1^0)^2 - z_2^0$ is

$$(-(\ln \Phi(t+t_0; \tau+\tau_0))', -(\ln \Phi(t+t_0; \tau+\tau_0))'', \wp(\tau+\tau_0), \wp'(\tau+\tau_0)).$$

For the coefficients of the Weierstrass σ-function

$$\sigma(u) = u \sum_{i,j\geq 0} \frac{a_{i,j}}{(4i+6j+1)!}(\frac{g_2u^4}{2})^i(2g_3u^6)^j,$$

there is the Weierstrass recursion

$$a_{i,j} = 3(i+1)a_{i+1,j-1} + \frac{16}{3}(j+1)a_{i-2,j+1} - \frac{1}{3}(4i+6j-1)(2i+3j-1)a_{i-1,j}$$

with the initial conditions

$$a_{0,0} = 1, \quad \text{and} \quad a_{i,j} = 0 \quad \text{for} \quad i < 0 \quad \text{or} \quad j < 0.$$

Conjecture. For $a(i,j) = 2^k 3^l s(i,j)$, and

$$\frac{(4i+6j+1)!}{2^{3i+4j}\, 3^{i+j}\, i!\, j!} = 2^{k_1}3^{l_1}s_1(i,j),$$

where $s(i,j)$ and $s_1(i,j) \in \mathbb{Z}$ are coprime with 2 and 3 we have $k = k_1$, $l = l_1$.

The Krichever genus.

I. M. Krichever introduced in 1990 the Hirzebruch genus L_f, where

$$f(t) = f_{Kr}(t) = \frac{\exp(\frac{\mu_1}{2}t)}{\Phi(t,\tau)},$$

and proved this genus to have the fundamental rigidity property on normally complex S^1-equivariant SU-manifolds.

Theorem. The series $f_{Kr}(t)$ is a Hurwits series over $\mathbb{Z}[\frac{\mu_1}{2}, \wp(\tau), \wp'(\tau), \frac{g_2}{2}]$, that is

$$f_{Kr}(t) = \sum_{k\geq 0} f_k \frac{t^{k+1}}{(k+1)!}, \qquad f_k \in \mathbb{Z}[\frac{\mu_1}{2}, \wp(\tau), \wp'(\tau), \frac{g_2}{2}].$$

The addition theorem for the Baker-Akhiezer function.

$$\Phi(t+q) = \frac{\Phi(t)\Phi'(q) - \Phi'(t)\Phi(q)}{\wp(t) - \wp(q)}.$$

For applications to the Kp hierarchy see in I.M. Krichever, FAA (1980).

Corrolary.

$$\frac{\Phi(t+q)}{\Phi(t)\Phi(q)} = -\frac{1}{2}\frac{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(\tau) \\ \wp'(t) & \wp'(q) & \wp'(\tau) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(\tau) \\ \wp(t)^2 & \wp(q)^2 & \wp(\tau)^2 \end{vmatrix}}.$$

The Krichever formal group.
Set $\mathcal{B} = \mathbb{Z}[\chi_k : k = 1, 2, ...]$.

Consider the series of the special form

$$\widehat{\mathcal{F}}(u,v) = ub(v) + vb(u) - b'(0)uv + \frac{b(u)\beta(u) - b(v)\beta(v)}{ub(v) - vb(u)}u^2v^2,$$

where $b(u) = 1 + \sum b_i u^i$, and $\beta(u) = \frac{b'(u)-b'(0)}{2u} = \sum_{k\geqslant 0} \beta_{k+2}u^k$.
Here $b_1 = \chi_1$, $b_{2i} = \chi_{2i}$ $b_{2i+1} = 2\chi_{2i+1}$ and
$\beta_{2k} = k\chi_{2k}$, $\beta_{2k+1} = (2k+ 1)\chi_{2k+1}$.

This series defines a formal group $F_{Kr}(u,v) \in \widehat{\mathcal{A}}[[u,v]]$
where $\widehat{\mathcal{A}} = \mathcal{B}/\widehat{\mathcal{J}}$ and $\widehat{\mathcal{J}}$ is the associativity ideal.

Theorem.

The exponential of the Krichever formal group is $f_{Kr}(t)$.

Theorem.
An elliptic formal group over the ring A with no zero divisors is a Krichever formal group if and only if in A we have:

$$\mu_2\mu_3 - \mu_1\mu_4 = 0, \qquad \mu_3^2 + 3\mu_6 = 0, \qquad \mu_3(\mu_1\mu_3 + \mu_4) = 0.$$

Corollary. The conditions of the theorem are equivalent to

$$\begin{aligned}
\mu_6 = 0 \qquad & \text{in the case } \mu_1 = 0, \mu_3 = 0;\\
\mu_2 = 0,\ \mu_3^2 = -3\mu_6, \qquad \mu_4 = 0 \qquad & \text{in the case } \mu_1 = 0, \mu_3 \neq 0;\\
\mu_4 = 0, \qquad \mu_6 = 0 \qquad & \text{in the case } \mu_1 \neq 0, \mu_3 = 0;\\
\mu_2 = -\mu_1^2,\ \mu_4 = -\mu_1\mu_3,\ -3\mu_6 = \mu_3^2 \qquad & \text{in the case } \mu_1 \neq 0, \mu_3 \neq 0.
\end{aligned}$$

Examples of elliptic Krichever formal groups (and integral Krichever genera).

Let $\mu_1 = \mu_3 = \mu_6 = 0$ and $\delta = \mu_2$, $\varepsilon = \mu_2^2 - 4\mu_4$, then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u\rho(v) + v\rho(u)}{1 - \varepsilon u^2 v^2}$$

for $\rho^2(u) = 1 - 2\delta u^2 + \varepsilon u^4$. In this case $f_{Kr}(t) = sn(t)$.

Let $\mu_1 = \mu_2 = \mu_4 = 0$ and $\mu_3^2 = -3\mu_6$, then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u^2 r(v) - v^2 r(u)}{u r^2(v) - v r^2(u)}$$

for $r^3(u) = 1 - 3\mu_3 u^3$.

Let $\mu_3 = \mu_4 = \mu_6 = 0$, then

$$F_{El}(u,v) = F_{Kr}(u,v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}\,.$$

The deformed Baker-Akhiezer function.

Let $f_{El}(t)$ be the exponential of $F_{El}(u,v)$. Then

$$\frac{1}{f_{El}(t)} - \frac{\mu_1}{2} = \phi(t;v,w) = -\frac{1}{2}\frac{\wp'(t)+\wp'(w)}{\wp(t)-\wp(v)}, \tag{2}$$

where $\wp(t) = \wp(t; g_2(\mu), g_3(\mu))$, $\wp'(w) = -\mu_3$, $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$.

Define *the deformed Baker-Akhiezer function* as the solution of

$$\Psi'(t) + \phi(t)\Psi(t) = 0$$

such that $\Psi(t) = \Psi(t;v,w) = 1/u +$ (regular function).

Lemma. For $\alpha = \frac{\wp'(w)}{\wp'(v)}$

$$\Psi(t) = \frac{\sigma(t+v)^{\frac{1}{2}(1-\alpha)}\sigma(v-t)^{\frac{1}{2}(1+\alpha)}}{\sigma(t)\sigma(v)} \exp\left((-\frac{\mu_1}{2} + \alpha\zeta(v))t\right).$$

The addition formula.

Theorem.

$$\Psi(t+q) =$$

$$= \frac{\begin{vmatrix} \Psi(t) & \Psi(q) \\ \Psi'(t) & \Psi'(q) \end{vmatrix}}{\wp(t) - \wp(q)} \times \frac{\begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(v) \\ \wp'(t) & \wp'(q) & \wp'(v) \end{vmatrix}^{\frac{1-\alpha}{2}} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(-v) \\ \wp'(t) & \wp'(q) & \wp'(-v) \end{vmatrix}^{\frac{1+\alpha}{2}}}{\frac{1-\alpha}{2} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(v) \\ \wp'(t) & \wp'(q) & \wp'(v) \end{vmatrix} + \frac{1+\alpha}{2} \begin{vmatrix} 1 & 1 & 1 \\ \wp(t) & \wp(q) & \wp(-v) \\ \wp'(t) & \wp'(q) & \wp'(-v) \end{vmatrix}}.$$

Properties of the deformed Baker-Akhiezer function for $\mu_1 = 0$.

1. In the vicinity of $t = 0$ we have

$$\Psi(t; v, w) = \frac{1}{t} - \frac{1}{2}\wp(v)t + \frac{1}{6}\wp'(w)t^2 + (t^3).$$

2. The function $\Psi(t; \pm v, w)$ gives a solution to the deformed Lame equation

$$\Psi''(t) - U\Psi(t) = \wp(v)\Psi(t),$$

$$\text{where } \; U = 2\wp(t) - \frac{\wp'(v)^2 - \wp'(w)^2}{4(\wp(t) - \wp(v))^2}.$$

3. The periodic properties are

$$\Psi(t+2\omega_k; v, w) = \Psi(t; v, w)\,\mathrm{exp}(2\alpha(\zeta(v)\omega_k - \eta_k v));$$

$$\Psi(t; v+2\omega_k, w) = \Psi(t; v, w).$$

The function $\Psi(t; \omega_k, w) = \Psi(t; \omega_k)$ does not depend on w and $\Psi(t+2\omega_k; \omega_k) = \Psi(t; \omega_k)$.

4. $\Psi(t; v, \omega_k) = \sqrt{\wp(t) - \wp(v)}$.

5. We have the relations

$$\Psi(t; v, w) = \Psi(t; -v, -w) = -\Psi(-t; v, -w).$$

Let $L_1^+ = L_1 = \partial + \phi(t)$ and $L_1^- = \partial - \phi(t)$ where

$$\phi(t) = -\frac{1}{2}\frac{\wp'(t) + \wp'(w)}{\wp(t) - \wp(v)}.$$

We have $L_2\Psi(t) = 0$, where $L_2 = \partial^2 - U - \wp(v) = L_1^- L_1^+$ and

$$U = 2\wp(t) - \frac{\wp'(v)^2 - \wp'(w)^2}{4(\wp(t) - \wp(v))^2}.$$

Set $V = \frac{(1-\alpha^2)}{16}\wp'(v)^2\mathcal{T}$, where $\mathcal{T} = \frac{(3\wp'(t)+\wp'(w))}{(\wp(t)-\wp(v))^3}$ и $(1-\alpha^2)\wp'(v)^2 = \wp'(v)^2 - \wp'(w)^2$.

The addition formula gives the operator $L_3 = 2\partial^3 - 3U\partial - U_0$, где $U_0 = \frac{3}{2}U' + 2V - \wp'(w)$, such that $L_3\Psi(t) = 0$. We have

$$[L_2, L_3] = -\frac{1}{4}(1-\alpha^2)\wp'(v)^2\left(\frac{\partial}{\partial t}\mathcal{T}\right)L_1.$$

The Hirzebruch genera.
Let $A = \sum A_{-2k}$ be a commutative associative graded torsion-free ring and let $f(u) = u + \sum_{k\geq 1} f_k u^{k+1}$, where $f_k \in A_{-2k} \otimes \mathbb{Q}$. Set

$$L_f(\sigma_1, ..., \sigma_k, ...) = \prod_{i=1}^{N} \frac{u_i}{f(u_i)},$$

where σ_k is the k-th elementary symmetric polynomial of $u_1, ..., u_N$.

We have

$$L_f(\sigma_1, ..., \sigma_n) = 1 - f_1\sigma_1 + (f_1^2 - f_2)\sigma_1^2 + (2f_2 - f_1^2)\sigma_2 + \dots$$

The Hirzebruch genus L_f of a stably complex manifold M^{2n} with tangent Chern classes $c_i = c_i(\tau(M^{2n}))$ and fundamental cycle $\langle M^{2n}\rangle$ is defined by the formula

$$L_f(M^{2n}) = (L_f(c_1, ..., c_n), \langle M^{2n}\rangle) \in A_{-2n} \otimes \mathbb{Q}.$$

Theorem.
The solution of the algebraic dynamical system on $\mathbb{C}^4$ with coordinates (z_1, z_2, x, y) (see below) is

$$(-(\ln\Phi(t+t_0;\tau+\tau_0))', -(\ln\Phi(t+t_0;\tau+\tau_0))'', \wp(t+t_0), \wp'(t+t_0)).$$

$$\begin{aligned}
\partial_t z_1 &= z_2, & \partial_\tau z_1 &= 3x + 2z_2 - z_1^2,\\
\partial_t z_2 &= 2z_1z_2 - 2y, & \partial_\tau z_2 &= -y + 2z_1z_2,\\
\partial_t x &= y, & \partial_\tau x &= 0,\\
\partial_t y &= -2z_2^2 + 2z_1^2z_2 - 2z_1y - 6xz_2, & \partial_\tau y &= 0.
\end{aligned}$$

$$\begin{aligned}
\partial_0 z_1 &= z_1, & \partial_2 z_1 &= z_1^3 - z_2z_1 - 4xz_1 - \frac{3}{2}y\\
\partial_0 z_2 &= 2z_2, & \partial_2 z_2 &= 4xz_2 + z_1z_2 - 2z_1^2z_2 + 2z_2^2\\
\partial_0 x &= 2x, & \partial_2 x &= -2x^2 - \frac{4}{3}z_2^2 + \frac{4}{3}z_1^2z_2 - \frac{4}{3}z_1y - 4xz_2\\
\partial_0 y &= 3y, & \partial_2 y &= 3xy
\end{aligned}$$