.

M a complete connected Riemannian manifold, $dimM = n$.

$G \subset Iso(M)$, closed and connected.

Isometric action of G on M

$$G \times M \to M$$

$$(g, x) \to gx$$

$x \in M$, the G-orbit containing x:

$$G(x) = \{gx : g \in G\}$$

Orbit space $\frac{M}{G} = \{G(x) : x \in M\}$

$$\pi : M \to \frac{M}{G}, \pi(x) = G(x)$$

If $\pi(x)$ is interior (boundary) point of $\frac{M}{G} \mapsto G(x)$ is called a principal (singular) orbit. $dim\frac{M}{G} = n - max_{x \in M} dimG(x)$

Definition: M is a G-manifold of cohomogeneity k if $dim\frac{M}{G} = k$. We denote it by $Coh(M, G) = k$.

Coh(M,G)=0

M **is homogeneous** $M \simeq \frac{G}{G_x}$

$$G_x = \{g \in G : gx = x\}$$

♣ **If** $Coh(M,G) = 0$ **and** $\kappa_M \leq 0 \Rightarrow M \simeq R^m \times T^{n-m}$ **(Wolf).**

Coh(M,G)=1

♣ $\frac{M}{G}$ **is homeomorphic to one of the following**

$$R, S^1, [0, +\infty), [-1, 1]$$

♣ **If** $\kappa_M < 0$**,** $dimM > 2$ **then**
$\pi_1(M) = 0$ **or** $\pi_1(M) = Z^p, p \geq 1$

If $p = 1 \Rightarrow$ **One orbit**$\simeq S^1$**; other orbits covered by** $S^{n-2} \times R$**.**
$p > 1 \Rightarrow$ **each orbit** $\simeq R^{n-1-p} \times T^p$**;** $M \simeq R^{n-p} \times T^p$**.**

$\kappa_M \leq 0$ **are also studied recently.**

$\kappa_M > 0$ **open problem.**

Coh(M,G)=2, $\kappa_M = 0$

Example: $M = R^{n-1} \times S^1$, $n \geq 3$, $G = SO(n-1)$

$$g \in SO(n-1), \;\; x = (x_1, x_2) \in R^{n-1} \times S^1 \Rightarrow g(x) = (gx_1, x_2)$$

Principal orbit = $S^{n-2}(c)$, for some c depending on orbits.
$x = (0, x_2) \in R^{n-1} \times S^1$, then the (singular) orbit $G(x)$ is equal to $\{x\}$.
The union of singualr orbits $\simeq S^1$.

Example Let $G_1 \subset SO(m-1)$, Coh$(G_1, S^{m-1}) = 1$ (for example $G_1 = SO(m-2)$).
Put $M^n = R^m \times T^{n-m}$, $n > m \geq 3$, $G = G_1 \times T^{n-m}$, which acts on $R^m \times T^{n-m}$ by product action.
Each principal G-orbit = $N^{m-2}(c) \times T^{n-m}$, where $N^{m-2}(c)$ is a homogeneous hypersurface of $S^{m-1}(c)$ (c depends on orbits).
If $x = (0, y) \in R^m \times T^{n-m}$ then the (singular) orbit $G(x)$ is isometric to T^{n-m}.

Example Let $M^n = R^3 \times T^{n-3}, n \geq 3$, and let $G = \{g_\theta = (e^{i\theta}, \theta) : \theta \in R\}$. Consider the following action of G on M:

$$x = (x_1, x_2, x_3) \in R^3, \;\; (x, y) \in R^3 \times T^{n-3}, \;\; g_\theta \in G$$

$$\Rightarrow g_\theta(x, y) = (x_1 \text{cos}\theta, x_2 \text{sin}\theta, x_3 + \theta, y)$$

M is a cohomogeneity two G-manifold and each principal orbit is isometric to $H \times T^{n-3}$, where H is a helix in R^3. If $x = (0, 0, x_3) \in R^3$ then the (singular) orbit $G(x, y)$ is isometric to $R \times T^{n-3}$.

Example Let $M^n = T^2 \times T^{n-2} \times R^m, n \geq 3$, and let $G = T^{n-2} \times R^m$, which acts on M in the following way:

$$g = (h, b) \in T^{n-2} \times R^m, \;\; (x_1, x_2, x_3) \in T^2 \times T^{n-2} \times R^m$$

$$\Rightarrow g(x) = (x_1, h(x_2), x_3 + b)$$

M is a cohomogeneity two G-manifold and each orbit is diffeomorphic to $T^{n-2} \times R^m$.

Theorem (Coh(M,G)=2, $\kappa_M = 0$) *One of the following is true:*

(a) M *is simply connected or* $\pi_1(M) = Z$
Each principal orbit=$S^{n-2}(c)$*, for some* $c > 0$ *(c depends on orbits).*

(b) $\pi_1(M) = Z^l$ *and one of the following is true:*
. **(b1)** *There is a positive integer* m, $2 < m < n$*, such that*
Each principal orbit is covered by $N^{m-2}(c) \times R^{n-m}$*, where* $N^{m-2}(c)$ *is a homogeneous hypersurface of* $S^{m-1}(c)$ *(* $c > 0$ *depends on orbits).*
There is a unique orbit diffeomorphic to $T^l \times R^{n-m-l}$*.*

. **(b2)** *Each principal orbit is covered by* $S^r \times R^{n-r-2}$*, for some positive integer* r*.*
. **(b3)** *Each principal orbit is covered by* $H \times R^{n-m}$*, such that* H *is a helix in* R^m*.*
There is an orbit diffeomorphic to $T^l \times R^t$*, for some non-negative integer* t*.*

(c) *Each orbit* $\simeq R^t \times T^l$*, for some nonnegative integer* t *(* $t = n - l - 2$*, if the orbit is principal)*

Theorem (Coh$(M^{n+2}, G) = 2$**,** $\kappa_M < 0$**,** $Fix(G, M) \neq \emptyset$**)**
Then

(a) M *is diffeomorphic to* $S^1 \times R^{n+1}$ *or* $B^2 \times R^n$*(* B^2 *is the mobius band).*

(b) $Fix(G, M)$ *is diffeomorphic to* S^1*.*

(c) Each principal orbit is diffeomorphic to S^n*.*

Theorem (Coh$(M^{n+2}, G) = 2$, $\kappa_M < 0$, G *is non-semisimple, singular orbits (if there is any) are fixed points of* G*)*
Then one of the following is true:

(1) M *is simply connected (diffeomorphic to* R^{n+2}*).*
(2) M *is diffeomorphic to* $S^1 \times R^{n+1}$ *or* $B^2 \times R^n$*(* B^2 *is the mobious band). Each principal orbit is diffeomorphic to* S^n*. Union of singular orbits (*$Fix(G, M)$*) is diffeomorphic to* S^1*.*
(3) M *is diffeomorphic to* $S^1 \times R^2$ *or* $B^2 \times R$*. All orbits are diffeomorphic to* S^1*.*
(4) $\pi_1(M) = Z^p$ *for some positive integer* p*, and all orbits are diffeomorphic to* $R^{n-p} \times T^p$*.*

Theorem (Coh$(M^{n+2}, G) = 2$**,** $\kappa_M = c < 0$**)**
Then either M *is simply connected or one of the following is true:*
(1) All orbits $\simeq T^{n-m} \times R^m$*.*
(2) $\pi_1(M) = Z$*, There is on orbit* $\simeq S^1$ *or* $Fix(G, M) = S^1$*.*
(3) $\pi_1(M) = Z^k$*,* $k > 1$*, and there is two types of orbits, one type diffeomorphic to* $T^k \times R^n$ *and the other types covered by* $S^{n-m} \times R^m$*.*

Theorem *If* R^n *,*$n \geq 3$*, is of cohomogeneity two under the action of a closed and connected Lie group* G *of isometries, then* $\frac{R^n}{G}$ *is homeomorphic to* R^2 *or* $[0, +\infty) \times R$*.*