# A Natural Geometric Framework for the Space of Initial Data of Nonlinear PDEs

Giovanni Moreno

University of Salerno    Levi-Civita Institute    INFN – GC Salerno

Geometry, Integrability and Quatization
Varna, Bulgaria
June 6–12, 2011

# Outline

1 Review of Jet Spaces and their Natural Structures

2 Jet Spaces of Pairs of Manifolds

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

**Field–Theoretic Setting.**

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

**Field–Theoretic Setting.**

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

## Field–Theoretic Setting.

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

## Field–Theoretic Setting.

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

## Field–Theoretic Setting.

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# Setting

Throughout the whole presentation,

$E$ is a fixed smooth manifold.

## Field–Theoretic Setting.

$E$ is the product $M \times T$, where

- $M$ is the space–time,
- and $T$ the target space of a field theory.

*In F. T. the number of independent variables can be 3 or 4, depending on the role of time.*

When $n$ is the number of independent variable, we set $m = \dim E - n$.

# 1–st Order Jet Spaces

## PHILOSOPHICAL DEFINITION

1–st order jet space of $E$:

"the smallest and smoothest container"

of all 1st–order approximations of all $n$–dimensional submanifolds $L \subseteq E$.

Symbol: $J^1(E, n)$.

# 1–st Order Jet Spaces

## PHILOSOPHICAL DEFINITION

1–st order jet space of $E$:

"the smallest and smoothest container"

of all 1st–order approximations of all $n$–dimensional submanifolds $L \subseteq E$.

Symbol: $J^1(E, n)$.

# 1–st Order Jet Spaces

## PHILOSOPHICAL DEFINITION

1–st order jet space of $E$:

“the smallest and smoothest container”

of all 1st–order approximations of all $n$–dimensional submanifolds $L \subseteq E$.

Symbol: $J^1(E, n)$.

Implementations:

- "1–st order approximation" $=$ $n$–dimensional linear subspace of $TE \Rightarrow J^1(E,n) = \mathrm{Gr}\,(TE,n)$;
- identify $n$–dimensional submanifolds tangent to each other $\Rightarrow$ $J^1(E,n) = \{\text{all submanifolds of dimension } n\}/\sim^1$;
- just add to the coordinates $(x^1,\ldots,x^n,u^1,\ldots,u^m)$ of $E$ new coordinates $u^j_i$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

## EXAMPLE: Field Theory

If $x^\mu$ is a point of the space–time, and $\phi^j$ the values of the field, the action of a 1–st order Lagrangian on $\phi$ is usually written as

$$S[\phi] = \int_M L(x^\mu, \phi^j, \phi^j_{(\mu)}).$$

Implementations:

- "1–st order approximation" $=$ $n$–dimensional linear subspace of $TE \Rightarrow J^1(E,n) = \mathrm{Gr}\,(TE,n)$;
- identify $n$–dimensional submanifolds tangent to each other $\Rightarrow$ $J^1(E,n) = \{\text{all submanifolds of dimension } n\}/\sim^1$;
- just add to the coordinates $(x^1,\ldots,x^n,u^1,\ldots,u^m)$ of $E$ new coordinates $u_i^j$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

## EXAMPLE: Field Theory

If $x^\mu$ is a point of the space–time, and $\phi^j$ the values of the field, the action of a 1–st order Lagrangian on $\phi$ is usually written as

$$S[\phi] = \int_M L(x^\mu, \phi^j, \phi^j_{(\mu)}).$$

# Projection

Canonical projection $J^1(E,n) \to E$ can be seen as:

- the bundle projection of $\mathrm{Gr}(TE,n)$ over $E$;
- the point of tangency of two submanifolds;
- the first $n+m$ coordinates of $J^1(E,n)$.

## NOTATION

- $L \sim^1_y L'$ is the tangency relation;
- $[L]^1_y$ the equivalence class of $L$;
- $\pi_{1,0}$ is the projection.

# Projection

Canonical projection $J^1(E,n) \to E$ can be seen as:

- the bundle projection of $\mathrm{Gr}\,(TE,n)$ over $E$;
- the point of tangency of two submanifolds;
- the first $n+m$ coordinates of $J^1(E,n)$.

NOTATION

- $L \sim_y^1 L'$ is the tangency relation;
- $[L]_y^1$ the equivalence class of $L$;
- $\pi_{1,0}$ is the projection.

# Projection

Canonical projection $J^1(E,n) \to E$ can be seen as:

- the bundle projection of $\mathrm{Gr}\,(TE,n)$ over $E$;
- the point of tangency of two submanifolds;
- the first $n+m$ coordinates of $J^1(E,n)$.

## NOTATION

- $L \sim^1_y L'$ is the tangency relation;
- $[L]^1_y$ the equivalence class of $L$;
- $\pi_{1,0}$ is the projection.

# Projection

Canonical projection $J^1(E,n) \to E$ can be seen as:

- the bundle projection of $\mathrm{Gr}(TE,n)$ over $E$;
- the point of tangency of two submanifolds;
- the first $n+m$ coordinates of $J^1(E,n)$.

**NOTATION**

- $L \sim_y^1 L'$ is the tangency relation;
- $[L]_y^1$ the equivalence class of $L$;
- $\pi_{1,0}$ is the projection.

# Projection

Canonical projection $J^1(E,n) \to E$ can be seen as:

- the bundle projection of $\mathrm{Gr}\,(TE,n)$ over $E$;
- the point of tangency of two submanifolds;
- the first $n+m$ coordinates of $J^1(E,n)$.

## NOTATION

- $L \sim_y^1 L'$ is the tangency relation;
- $[L]_y^1$ the equivalence class of $L$;
- $\pi_{1,0}$ is the projection.

# R–distribution

To any point

$$\theta = [L]_y^1 \in J^1(E, n)$$

associate the linear subspace

$$R_\theta = T_y L \leq T_y E$$

.

**DEFINITION**

$R$ is the canonical (relative to $\pi_{1,0}$) distribution on $E$.

# R–distribution

To any point

$$\theta = [L]^1_y \in J^1(E, n)$$

associate the linear subspace

$$R_\theta = T_y L \leq T_y E$$

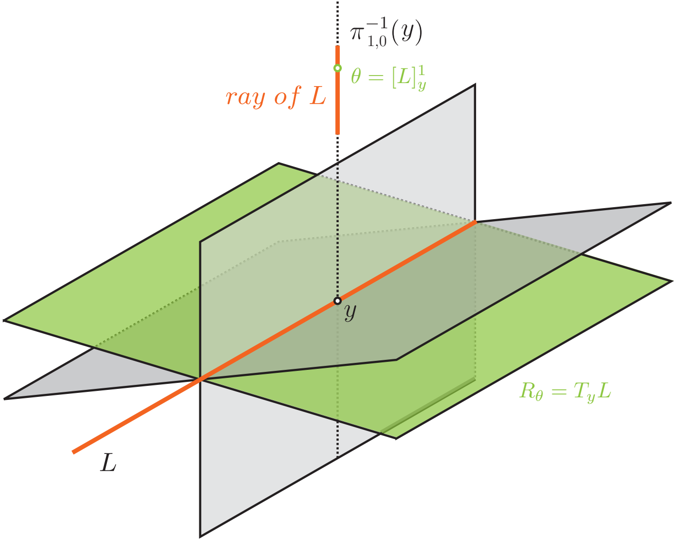
.

## DEFINITION

$R$ is the canonical (relative to $\pi_{1,0}$) distribution on $E$.

# Ray manifolds

Of particular interest (e.g., Lie–Bäcklund theorem) are the so–called ray manifolds.



# Jet–prolongation of submanifolds

The embedding

$$L \subseteq E$$

is canonically lifted to an embedding

$$j_1(L) : L \longrightarrow J^1(E, n),$$

where

$$j_1(L)(y) \overset{\text{def}}{=} [L]^1_y.$$

**DEFINITION**

$j_1(L)$ is the 1st jet–prolongation of $L$.
Its image is denoted by $L^{(1)}$.

# Jet–prolongation of submanifolds

The embedding

$$L \subseteq E$$

is canonically lifted to an embedding

$$j_1(L) : L \longrightarrow J^1(E, n),$$

where

$$j_1(L)(y) \overset{\text{def}}{=} [L]^1_y.$$

**DEFINITION**

$j_1(L)$ is the 1st jet–prolongation of $L$.
Its image is denoted by $L^{(1)}$.

# Jet–prolongation of submanifolds

The embedding

$$L \subseteq E$$

is canonically lifted to an embedding

$$j_1(L) : L \longrightarrow J^1(E, n),$$

where

$$j_1(L)(y) \stackrel{\text{def}}{=} [L]^1_y.$$

## DEFINITION

$j_1(L)$ is the 1st jet–prolongation of $L$.
Its image is denoted by $L^{(1)}$.

# Differential equations

A submanifold

$$\mathcal{E} \subseteq J^1(E, n),$$

with

$$\mathcal{E} : F^\alpha = 0$$

is interpreted as a (system of) 1–st order nonlinear PDE(s).

**DEFINITION**

$L$ is a solution of $\mathcal{E}$ iff

$$L^{(1)} \subseteq \mathcal{E},$$

or, equivalently, if $j_1(L)^*(F^\alpha) = 0$, for all $\alpha$'s.

# Differential equations

A submanifold

$$\mathcal{E} \subseteq J^1(E, n),$$

with

$$\mathcal{E} : F^\alpha = 0$$

is interpreted as a (system of) 1–st order nonlinear PDE(s).

**DEFINITION**

$L$ is a solution of $\mathcal{E}$ iff

$$L^{(1)} \subseteq \mathcal{E},$$

or, equivalently, if $j_1(L)^*(F^\alpha) = 0$, for all $\alpha$'s.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u^j_i)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]^1_y}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u^j_i)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]^1_y}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u_i^j)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u_i^j$, $(u_i^j)_k = (u_k^j)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]_y^1}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u^j_i)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]^1_y}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u^j_i)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]^1_y}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Higher–order jets

Passing from $J^1(E,n)$ to

$$J^1(J^1(E,n),n)$$

means adding new coordinates $(u^j)_i$ and $(u^j_i)_k$.

## DEFINITION

In $J^1(J^1(E,n),n)$ there lives the distinguished subset $J^2(E,n)$, which may be thought of as:

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$;
- the jets of olonomic submanifolds $[L^{(1)}]_{[L]^1_y}$ of $J^1(E,n)$;
- the set of $R$–horizontal $n$–dimensional planes of $J^1(E,n)$, i.e., $\Theta_\theta$ such that $d\pi_{1,0}(R_\Theta) = R_\theta$,

and it is called the 2–nd order jet space, and identifies with the quotient space of all submanifolds modulo 2nd–order tangency.

# Cartan distribution

Along the projection

$$\pi_{2,1} = \pi_{1,0}|_{J^2(E,n)}$$

there is the relative distribution $R$.
All the $R$–planes passing trough $\theta$ generate the Cartan plane $\mathcal{C}_\theta$.
It is easy to see that

$$\mathcal{C}_\theta = R_\theta \oplus T(\pi_{1,0}^{-1}(\pi_{1,0}(\theta))).$$

**THEOREM**

Jet–prolongations $L^{(1)}$ are precisely the maximal $\pi_{(1,0)}$–horizontal integral submanifolds of $\mathcal{C}$.
Other integral submanifolds are the ray–manifolds.

## Cartan distribution

Along the projection

$$\pi_{2,1} = \pi_{1,0}|_{J^2(E,n)}$$

there is the relative distribution $R$.
All the $R$–planes passing trough $\theta$ generate the Cartan plane $\mathcal{C}_\theta$.
It is easy to see that

$$\mathcal{C}_\theta = R_\theta \oplus T(\pi_{1,0}^{-1}(\pi_{1,0}(\theta))).$$

**THEOREM**

Jet–prolongations $L^{(1)}$ are precisely the maximal $\pi_{(1,0)}$–horizontal integral submanifolds of $\mathcal{C}$.
Other integral submanifolds are the ray–manifolds.

# Cartan distribution

Along the projection

$$\pi_{2,1} = \pi_{1,0}|_{J^2(E,n)}$$

there is the relative distribution $R$.
All the $R$–planes passing trough $\theta$ generate the Cartan plane $\mathcal{C}_\theta$.
It is easy to see that

$$\mathcal{C}_\theta = R_\theta \oplus T(\pi_{1,0}^{-1}(\pi_{1,0}(\theta))).$$

**THEOREM**

Jet–prolongations $L^{(1)}$ are precisely the maximal $\pi_{(1,0)}$–horizontal integral submanifolds of $\mathcal{C}$.
Other integral submanifolds are the ray–manifolds.

# Cartan distribution

Along the projection

$$\pi_{2,1} = \pi_{1,0}|_{J^2(E,n)}$$

there is the relative distribution $R$.
All the $R$–planes passing trough $\theta$ generate the Cartan plane $\mathcal{C}_\theta$.
It is easy to see that

$$\mathcal{C}_\theta = R_\theta \oplus T(\pi_{1,0}^{-1}(\pi_{1,0}(\theta))).$$

**THEOREM**

Jet–prolongations $L^{(1)}$ are precisely the maximal $\pi_{(1,0)}$–horizontal integral submanifolds of $\mathcal{C}$.
Other integral submanifolds are the ray–manifolds.

# Infinite jet spaces

The limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

gives $J^\infty(E,n)$. The Cartan distribution becomes completely integrable and $n$–dimensional!

**THEOREM**

Jet–prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.

**Coordinates.**

The distribution $\mathcal{C}$ can be equivalently defined as

- an infinite Pfaff system $\omega^j_\sigma = 0$ (in F.T. the $\omega^j_\sigma$'s look like $\omega^j_\mu = d_V \phi^j_{(\mu)}$);
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma} \qquad 1_i = \underbrace{(0,\ldots,1,\ldots,0)}_{i\text{–th place}}$$

# Infinite jet spaces

The limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

gives $J^\infty(E,n)$. The Cartan distribution becomes completely integrable and $n$–dimensional!

**THEOREM**

Jet–prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.

**Coordinates.**

The distribution $\mathcal{C}$ can be equivalently defined as

- an infinite Pfaff system $\omega^j_\sigma = 0$ (in F.T. the $\omega^j_\sigma$'s look like $\omega^j_\mu = d_V\phi^j_{(\mu)}$);
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i}\frac{\partial}{\partial u^j_\sigma} \qquad 1_i = (0,\ldots,\underbrace{1}_{i\text{–th place}},\ldots,0)$$

# Infinite jet spaces

The limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

gives $J^\infty(E,n)$. The Cartan distribution becomes completely integrable and $n$–dimensional!

## THEOREM

Jet–prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.

## Coordinates.

The distribution $\mathcal{C}$ can be equivalently defined as

- an infinite Pfaff system $\omega_\sigma^j = 0$ (in F.T. the $\omega_\sigma^j$'s look like $\omega_\mu^j = d_V \phi_{(\mu)}^j$);
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u_{\sigma+1_i}^j \frac{\partial}{\partial u_\sigma^j} \qquad 1_i = (0,\ldots,\underbrace{1}_{i\text{–th place}},\ldots,0)$$

# Infinite jet spaces

The limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

gives $J^\infty(E,n)$. The Cartan distribution becomes completely integrable and $n$–dimensional!

## THEOREM

Jet–prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.

## Coordinates.

The distribution $\mathcal{C}$ can be equivalently defined as

- an infinite Pfaff system $\omega_\sigma^j = 0$ (in F.T. the $\omega_\sigma^j$'s look like $\omega_\mu^j = d_V \phi_{(\mu)}^j$);
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma} \quad 1_i = (0,\ldots,\underset{i\text{–th place}}{1},\ldots,0)$$

# Infinite jet spaces

The limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

gives $J^\infty(E,n)$. The Cartan distribution becomes completely integrable and $n$–dimensional!

## THEOREM

Jet–prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.

## Coordinates.

The distribution $\mathcal{C}$ can be equivalently defined as

- an infinite Pfaff system $\omega^j_\sigma = 0$ (in F.T. the $\omega^j_\sigma$'s look like $\omega^j_\mu = d_V \phi^j_{(\mu)}$);
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i} \frac{\partial}{\partial u^j_\sigma} \qquad 1_i = (0,\ldots,\underset{i\text{–th place}}{1},\ldots,0)$$

# INFINITELY PROLONGED EQUATIONS

## A PROBLEM?

The restricted distribution $\mathcal{C}|_{\mathcal{E}}$ is not completely integrable, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$!

The biggest submanifold of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the

infinite prolongation of $\mathcal{E}$

and denoted by

$$\mathcal{E}_\infty.$$

Algebraically the latter is obtained from the former by adding to the $F^\alpha$'s all their differential consequences (i.e., the total derivatives).

# INFINITELY PROLONGED EQUATIONS

## A PROBLEM?

The restricted distribution $\mathcal{C}|_{\mathcal{E}}$ is not completely integrable, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$!

The biggest submanifold of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the

infinite prolongation of $\mathcal{E}$

and denoted by

$$\mathcal{E}_\infty.$$

Algebraically the latter is obtained from the former by adding to the $F^\alpha$'s all their differential consequences (i.e., the total derivatives).

# INFINITELY PROLONGED EQUATIONS

## A PROBLEM?

The restricted distribution $\mathcal{C}|_{\mathcal{E}}$ is not completely integrable, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$!

The biggest submanifold of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the

infinite prolongation of $\mathcal{E}$

and denoted by

$$\mathcal{E}_\infty.$$

Algebraically the latter is obtained from the former by adding to the $F^\alpha$'s all their differential consequences (i.e., the total derivatives).

# INFINITELY PROLONGED EQUATIONS

## A PROBLEM?

The restricted distribution $\mathcal{C}|_{\mathcal{E}}$ is not completely integrable, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$!

The biggest submanifold of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the

infinite prolongation of $\mathcal{E}$

and denoted by

$$\mathcal{E}_\infty .$$

Algebraically the latter is obtained from the former by adding to the $F^\alpha$'s all their differential consequences (i.e., the total derivatives).

## Example.

In F.T. the differential consequences of $F = 0$ are denoted by $\partial_{(\mu)} F = 0$. For example,

$$\partial_{(\mu)} \frac{\delta L}{\delta \phi^i} = 0$$

represent the infinitely prolonged Euler–Lagrange equations associated with a Lagrangian $L$, i.e., the Covariant Phase Space associated with $L$.

## Example.

In F.T. the differential consequences of $F=0$ are denoted by $\partial_{(\mu)}F=0$. For example,

$$\partial_{(\mu)}\frac{\delta L}{\delta\phi^i}=0$$

represent the infinitely prolonged Euler–Lagrange equations associated with a Lagrangian $L$, i.e., the Covariant Phase Space associated with $L$.

# Diffieties

A diffiety (from differential variety) is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite–dimensional completely integrable distribution on it. Leaves of $\mathcal{C}$ are called the secondary points of the diffiety, and their totality can be denoted by $M$.

## EXAMPLES

- If $\mathcal{O}$ is a fiber bundle, and $\mathcal{C}$ is the vertical distribution on it, then $M$ is just the base of the bundle (i.e., the manifold of all the fibers)!
- $(\mathcal{E}_\infty, \mathcal{C}|_{\mathcal{E}_\infty})$ is a diffiety, and $M$ is precisely the set of solutions of $\mathcal{E}$.

Accordingly to personal tastes, modifier secondary can be replaced by variational or functional.

# Diffieties

A diffiety (from differential variety) is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite–dimensional completely integrable distribution on it. Leaves of $\mathcal{C}$ are called the secondary points of the diffiety, and their totality can be denoted by $\boldsymbol{M}$.

## EXAMPLES

- If $\mathcal{O}$ is a fiber bundle, and $\mathcal{C}$ is the vertical distribution on it, then $\boldsymbol{M}$ is just the base of the bundle (i.e., the manifold of all the fibers)!
- $(\mathcal{E}_\infty, \mathcal{C}|_{\mathcal{E}_\infty})$ is a diffiety, and $\boldsymbol{M}$ is precisely the set of solutions of $\mathcal{E}$.

Accordingly to personal tastes, modifier secondary can be replaced by variational or functional.

# Diffieties

A diffiety (from differential variety) is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite–dimensional completely integrable distribution on it. Leaves of $\mathcal{C}$ are called the secondary points of the diffiety, and their totality can be denoted by $\boldsymbol{M}$.

## EXAMPLES

- If $\mathcal{O}$ is a fiber bundle, and $\mathcal{C}$ is the vertical distribution on it, then $\boldsymbol{M}$ is just the base of the bundle (i.e., the manifold of all the fibers)!
- $(\mathcal{E}_\infty, \mathcal{C}|_{\mathcal{E}_\infty})$ is a diffiety, and $\boldsymbol{M}$ is precisely the set of solutions of $\mathcal{E}$.

Accordingly to personal tastes, modifier secondary can be replaced by variational or functional.

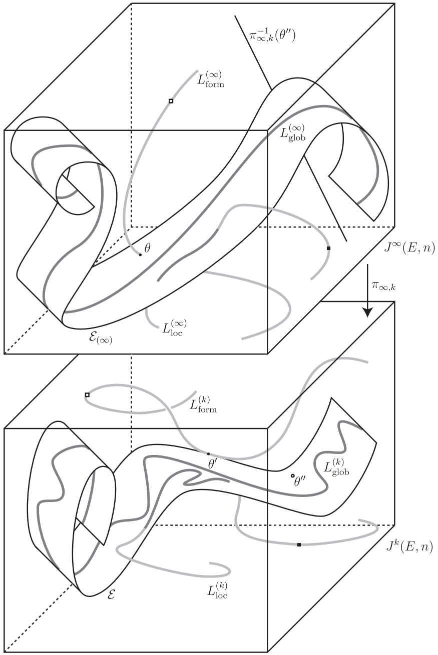
# Diffieties

A diffiety (from differential variety) is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite–dimensional completely integrable distribution on it. Leaves of $\mathcal{C}$ are called the secondary points of the diffiety, and their totality can be denoted by $\boldsymbol{M}$.

## EXAMPLES

- If $\mathcal{O}$ is a fiber bundle, and $\mathcal{C}$ is the vertical distribution on it, then $\boldsymbol{M}$ is just the base of the bundle (i.e., the manifold of all the fibers)!
- $(\mathcal{E}_\infty, \mathcal{C}|_{\mathcal{E}_\infty})$ is a diffiety, and $\boldsymbol{M}$ is precisely the set of solutions of $\mathcal{E}$.

Accordingly to personal tastes, modifier secondary can be replaced by variational or functional.


$\pi^{-1}_{\infty,k}(\theta'')$
$L^{(\infty)}_{\mathrm{form}}$
$L^{(\infty)}_{\mathrm{glob}}$
$\theta$
$J^\infty(E,n)$
$\pi_{\infty,k}$
$L^{(\infty)}_{\mathrm{loc}}$
$\mathcal{E}_{(\infty)}$
$L^{(k)}_{\mathrm{form}}$
$\theta'$
$\theta''$
$L^{(k)}_{\mathrm{glob}}$
$J^k(E,n)$
$\mathcal{E}$
$L^{(k)}_{\mathrm{loc}}$

# ELEMENTS OF SECONDARY CALCULUS

## SECONDARY VECTOR FIELDS ON $\mathcal{E}_\infty$ ARE INFINITESIMAL SYMMETRIES OF $\mathcal{E}$

**Vector fields "respecting" $\mathcal{C}|_{\mathcal{E}_\infty}$ are contact fields**

$$D_{\mathcal{E}} = \{\text{vector fields } X \text{ on } \mathcal{E}_\infty \text{ such that } [X, \overline{D}_i] = \sum \phi_j \overline{D}_j\},$$

where $\overline{D}_i$ is the restriction of $D_i$ to $\mathcal{E}_\infty$.

$X \in D_{\mathcal{E}}$ sends solutions to solutions.
$X, Y \in D_{\mathcal{E}}$ are equivalent
if they generate the same flow in the space of solutions of $\mathcal{E}$.

# ELEMENTS OF SECONDARY CALCULUS

SECONDARY VECTOR FIELDS ON $\mathcal{E}_\infty$
ARE INFINITESIMAL SYMMETRIES OF $\mathcal{E}$

### Vector fields "respecting" $\mathcal{C}|_{\mathcal{E}_\infty}$ are contact fields

$$D_{\mathcal{E}} = \{\text{vector fields } X \text{ on } \mathcal{E}_\infty \text{ such that } [X, \overline{D}_i] = \sum \phi_j \overline{D}_j\},$$

where $\overline{D}_i$ is the restriction of $D_i$ to $\mathcal{E}_\infty$.

$X \in D_{\mathcal{E}}$ sends solutions to solutions.
$X, Y \in D_{\mathcal{E}}$ are equivalent
if they generate the same flow in the space of solutions of $\mathcal{E}$.

# ELEMENTS OF SECONDARY CALCULUS

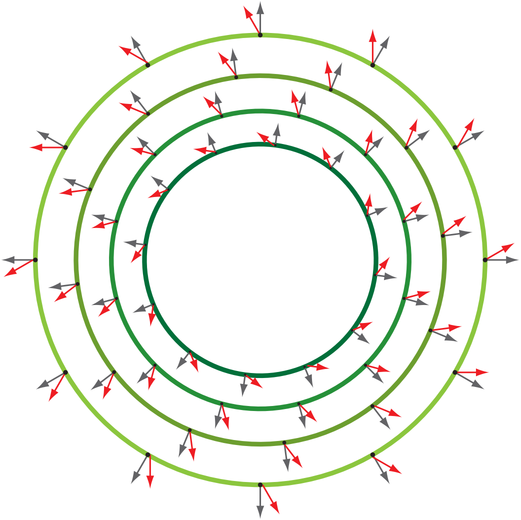
SECONDARY VECTOR FIELDS ON $\mathcal{E}_\infty$
ARE INFINITESIMAL SYMMETRIES OF $\mathcal{E}$

### Vector fields "respecting" $\mathcal{C}|_{\mathcal{E}_\infty}$ are contact fields

$$D_{\mathcal{E}} = \{\text{vector fields } X \text{ on } \mathcal{E}_\infty \text{ such that } [X, \overline{D}_i] = \sum \phi_j \overline{D}_j\},$$

where $\overline{D}_i$ is the restriction of $D_i$ to $\mathcal{E}_\infty$.

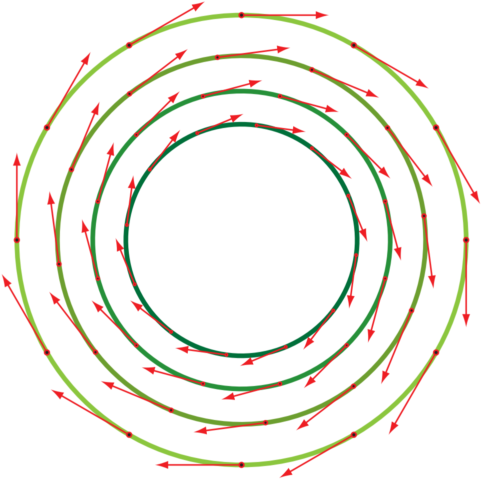
$X \in D_{\mathcal{E}}$ sends solutions to solutions.
$X, Y \in D_{\mathcal{E}}$ are equivalent
if they generate the same flow in the space of solutions of $\mathcal{E}$.



black and red are equivalent

# TRIVIAL CONTACT FIELDS

$$\mathcal{C}D_{\mathcal{E}} = \{X = \sum f_i \overline{D}_i\}$$



$$X \sim Y \Leftrightarrow X - Y \in \mathcal{C}D(\mathcal{E})$$

$$\boxed{\operatorname{sym}\mathcal{E} = \frac{D_{\mathcal{C}}(\mathcal{E})}{\mathcal{C}D(\mathcal{E})}}$$

HIGHER SYMMETRIES OF $\mathcal{E}$

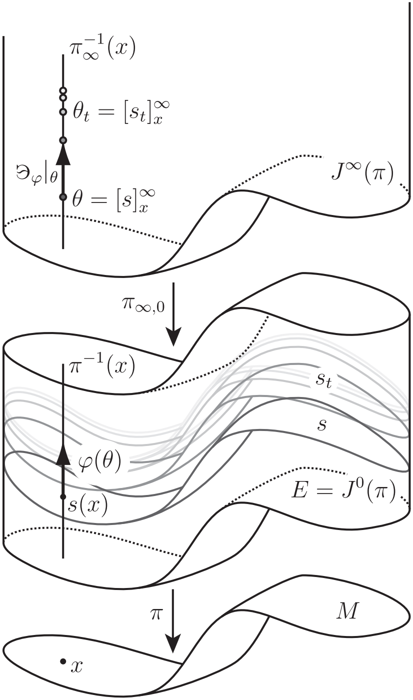
$$\boxed{\operatorname{sym}\mathcal{E} = H^0(\text{Horizontal Jet Spencer Complex on } \mathcal{E}_{(\infty)})}$$

## SECONDARY VECTOR FIELDS ON $J^\infty$

$$\varkappa = \mathrm{sym}\, J^\infty = \{\ni_\varphi \mid \varphi = (\varphi_1, \dots, \varphi_m), \quad \varphi_i \in C^\infty(J^\infty)\}$$

$$\ni_\varphi \overset{\mathrm{def}}{=} \sum_{\sigma, i} D_\sigma(\varphi_i) \frac{\partial}{\partial u^i_\sigma}, \quad D_\sigma = D_1^{\sigma_1} \circ \cdots D_n^{\sigma_n}$$

$\varphi$ is generating function of $\chi = \ni_\varphi \mod \mathcal{C}D(J^\infty)$


$\pi_\infty^{-1}(x)$
$\theta_t = [s_t]_x^\infty$
$\ni_\varphi|_\theta$
$\theta = [s]_x^\infty$
$J^\infty(\pi)$
$\pi_{\infty,0}$
$\pi^{-1}(x)$
$s_t$
$s$
$\varphi(\theta)$
$s(x)$
$E = J^0(\pi)$
$\pi$
$M$
$x$

# HORIZONTAL COHOMOLOGY

$$\overline{\Lambda}^i(\mathcal{E}_\infty) \stackrel{\text{def}}{=} \frac{\Lambda^i(\mathcal{E}_\infty)}{\mathcal{C}\Lambda^i(\mathcal{E}_\infty)}, \quad \bar{d} : \overline{\Lambda}^i \to \overline{\Lambda}^{i+1},$$

where $\mathcal{C}\Lambda(\mathcal{E}_\infty)$ is the ideal of the differential forms vanishing on the Cartan distribution.

Horizontal de Rham complex of $\mathcal{E}_\infty$:

$$0 \to \overline{\Lambda}^0(\mathcal{E}_\infty) = C^\infty(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \overline{\Lambda}^1(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \overline{\Lambda}^n(\mathcal{E}_\infty) \to 0$$

Cohomologies of this complex $\overline{H}^i(\mathcal{E}_\infty)$ are called horizontal.

## INTERESTING TERMS:

- Lagrangians: $\overline{H}^n(J^\infty(E,n))$;
- Conservation laws: $\overline{H}^{n-1}(\mathcal{E}_\infty)$.

# HORIZONTAL COHOMOLOGY

$$\overline{\Lambda}^i(\mathcal{E}_\infty) \overset{\text{def}}{=} \frac{\Lambda^i(\mathcal{E}_\infty)}{\mathcal{C}\Lambda^i(\mathcal{E}_\infty)}, \quad \bar{d} : \overline{\Lambda}^i \to \overline{\Lambda}^{i+1},$$

where $\mathcal{C}\Lambda(\mathcal{E}_\infty)$ is the ideal of the differential forms vanishing on the Cartan distribution.

Horizontal de Rham complex of $\mathcal{E}_\infty$:

$$0 \to \overline{\Lambda}^0(\mathcal{E}_\infty) = C^\infty(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \overline{\Lambda}^1(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \overline{\Lambda}^n(\mathcal{E}_\infty) \to 0$$

Cohomologies of this complex $\overline{H}^i(\mathcal{E}_\infty)$ are called horizontal.

## INTERESTING TERMS:

- Lagrangians: $\overline{H}^n(J^\infty(E,n))$;
- Conservation laws: $\overline{H}^{n-1}(\mathcal{E}_\infty)$.

# HORIZONTAL COHOMOLOGY

$$\overline{\Lambda}^i(\mathcal{E}_\infty) \stackrel{\text{def}}{=} \frac{\Lambda^i(\mathcal{E}_\infty)}{\mathcal{C}\Lambda^i(\mathcal{E}_\infty)}, \quad \bar{d} : \overline{\Lambda}^i \to \overline{\Lambda}^{i+1},$$

where $\mathcal{C}\Lambda(\mathcal{E}_\infty)$ is the ideal of the differential forms vanishing on the Cartan distribution.
Horizontal de Rham complex of $\mathcal{E}_\infty$:

$$0 \to \overline{\Lambda}^0(\mathcal{E}_\infty) = C^\infty(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \overline{\Lambda}^1(\mathcal{E}_\infty) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \overline{\Lambda}^n(\mathcal{E}_\infty) \to 0$$

Cohomologies of this complex $\overline{H}^i(\mathcal{E}_\infty)$ are called horizontal.

## INTERESTING TERMS:

- Lagrangians: $\overline{H}^n(J^\infty(E, n))$;
- Conservation laws: $\overline{H}^{n-1}(\mathcal{E}_\infty)$.

# $\mathcal{C}$-SPECTRAL SEQUENCES

Take now the powers of $\mathcal{C}\Lambda^*(\mathcal{E}_\infty)$:

filtered complex: $\Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}\Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}_\infty) \supset \cdots$

$\Downarrow$

the associated spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ is called $\mathcal{C}$–spectral

## INTERESTING TERMS:

- Euler operator: $d_1^{0,n}$;
- LHS of E–L equations: $E_1^{1,n}$;
- Helmholtz conditions: $E_1^{2,n}$

# $\mathcal{C}$-SPECTRAL SEQUENCES

Take now the powers of $\mathcal{C}\Lambda^*(\mathcal{E}_\infty)$:

$$\text{filtered complex: } \Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}\Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}_\infty) \supset \cdots$$

$$\Downarrow$$

the associated spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ is called $\underline{\mathcal{C}\text{–spectral}}$

## INTERESTING TERMS:

- Euler operator: $d_1^{0,n}$;
- LHS of E–L equations: $E_1^{1,n}$;
- Helmholtz conditions: $E_1^{2,n}$

# $\mathcal{C}$-SPECTRAL SEQUENCES

Take now the powers of $\mathcal{C}\Lambda^*(\mathcal{E}_\infty)$:

$$\text{filtered complex: } \Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}\Lambda^*(\mathcal{E}_\infty) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}_\infty) \supset \cdots$$

$$\Downarrow$$

the associated spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ is called $\mathcal{C}$–spectral

## INTERESTING TERMS:

- Euler operator: $d_1^{0,n}$;
- LHS of E–L equations: $E_1^{1,n}$;
- Helmholtz conditions: $E_1^{2,n}$

# Cohomology vs. Topology

# Outline

1. Review of Jet Spaces and their Natural Structures

2. Jet Spaces of Pairs of Manifolds

# Definition

Now we fix two numbers $n_2 \geq n_1$, and $k \geq l$.
Define

$$J^{k,l}(E, n_2, n_1)$$

as the subset of

$$J^k(E, n_2) \times_E J^l(E, n_1)$$

made by those elements $([L_2]^k_y, [L_1]^l_y)$ such that

$$L_2 \sim^k_y L_1 .$$

# Definition

Now we fix two numbers $n_2 \geq n_1$, and $k \geq l$.
Define

$$J^{k,l}(E, n_2, n_1)$$

as the subset of

$$J^k(E, n_2) \times_E J^l(E, n_1)$$

made by those elements $([L_2]^k_y, [L_1]^l_y)$ such that

$$L_2 \sim^k_y L_1.$$

# Definition

Now we fix two numbers $n_2 \geq n_1$, and $k \geq l$.
Define

$$J^{k,l}(E, n_2, n_1)$$

as the subset of

$$J^k(E, n_2) \times_E J^l(E, n_1)$$

made by those elements $([L_2]^k_y, [L_1]^l_y)$ such that

$$L_2 \sim^k_y L_1.$$

# Flag bundles

This construction is just a differential generalization of a well–known concept: the flag!

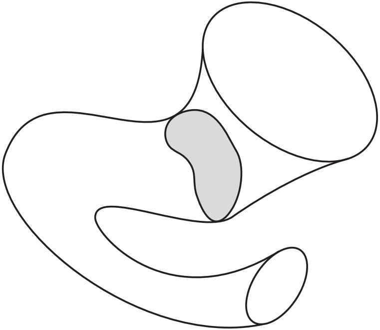
$$J^{1,1}(E, n_2, n_1) = \mathrm{Gr}\left(TE, n_2, n_1\right)$$

# Normal bundles

The bundle

$$\nu_\infty : J^{\infty,1}(E, n, n-1) \longrightarrow J^1(E, n-1)$$

naturally interpreted as the *normal bundle*, is fundamental in the cohomological approach to natural boundary conditions.



G. Moreno, *GIQ Proceedings, 2009*

G. Moreno and A. Vinogradov, *Doklady Mathematics, 2007*

# Canonical embedding into iterated jet spaces

## LEMMA

$J^{\infty,\infty}(E, n_2, n_1)$ is embedded canonically into $J^{\infty}(J^{\infty}(E, n_2), n_1)$.

Indeed, for any point $([L_2]^{\infty}_y, [L_1]^{\infty}_y)$ the jet prolongation

$$j_{\infty}(L_2) : L_2 \longrightarrow J^{\infty}(E, n_2)$$

can be used to lift $L_1$ inside $J^{\infty}(E, n_2)$.

So we obtain the $n_1$–dimensional submanifold $(j_{\infty}(L_2))(L_1)$, of which we can take the $\infty$–jet at the point $[L_2]^{\infty}_y$:

$$([L_2]^{\infty}_y, [L_1]^{\infty}_y) \longmapsto [(j_{\infty}(L_2))(L_1)]_{[L_2]^{\infty}_y}$$

# Canonical embedding into iterated jet spaces

## LEMMA

$J^{\infty,\infty}(E, n_2, n_1)$ is embedded canonically into $J^\infty(J^\infty(E, n_2), n_1)$.

Indeed, for any point $([L_2]^\infty_y, [L_1]^\infty_y)$ the jet prolongation

$$j_\infty(L_2) : L_2 \longrightarrow J^\infty(E, n_2)$$

can be used to lift $L_1$ inside $J^\infty(E, n_2)$.

So we obtain the $n_1$–dimensional submanifold $(j_\infty(L_2))(L_1)$, of which we can take the $\infty$–jet at the point $[L_2]^\infty_y$:

$$([L_2]^\infty_y, [L_1]^\infty_y) \longmapsto [(j_\infty(L_2))(L_1)]_{[L_2]^\infty_y}$$

# Canonical embedding into iterated jet spaces

**LEMMA**

$J^{\infty,\infty}(E, n_2, n_1)$ is embedded canonically into $J^{\infty}(J^{\infty}(E, n_2), n_1)$.

Indeed, for any point $([L_2]^{\infty}_y, [L_1]^{\infty}_y)$ the jet prolongation

$$j_{\infty}(L_2) : L_2 \longrightarrow J^{\infty}(E, n_2)$$

can be used to lift $L_1$ inside $J^{\infty}(E, n_2)$.
So we obtain the $n_1$–dimensional submanifold $(j_{\infty}(L_2))(L_1)$, of which we can take the $\infty$–jet at the point $[L_2]^{\infty}_y$:

$$([L_2]^{\infty}_y, [L_1]^{\infty}_y) \longmapsto [(j_{\infty}(L_2))(L_1)]_{[L_2]^{\infty}_y}$$

# Canonical embedding into iterated jet spaces

## LEMMA

$J^{\infty,\infty}(E,n_2,n_1)$ is embedded canonically into $J^\infty(J^\infty(E,n_2),n_1)$.

Indeed, for any point $([L_2]_y^\infty,[L_1]_y^\infty)$ the jet prolongation

$$j_\infty(L_2):L_2\longrightarrow J^\infty(E,n_2)$$

can be used to lift $L_1$ inside $J^\infty(E,n_2)$.
So we obtain the $n_1$–dimensional submanifold $(j_\infty(L_2))(L_1)$, of which we can take the $\infty$–jet at the point $[L_2]_y^\infty$:

$$([L_2]_y^\infty,[L_1]_y^\infty)\longmapsto[(j_\infty(L_2))(L_1)]_{[L_2]_y^\infty}$$

# The space of initial data

In the case $n_2 = n$ and $n_1 = n - 1$, then

$$\boxed{\mathcal{E} = \{(\phi^j_{(\sigma+l_n)})_{(\mu)} = \phi^j_{(\sigma+1_\mu+l_n)} + \phi^j_{\sigma+(l+1)_n} t_{(\mu)}\}_\infty}$$

is the defining equation of $J^{\infty,\infty}(E, n, n-1)$.

**DEFINITION**

$J^{\infty,\infty}(E, n, n-1)$ is the diffiety of initial data.

# The space of initial data

In the case $n_2 = n$ and $n_1 = n - 1$, then

$$\boxed{\mathcal{E} = \{(\phi^j_{(\sigma+l_n)})_{(\mu)} = \phi^j_{(\sigma+1_\mu+l_n)} + \phi^j_{\sigma+(l+1)_n} t_{(\mu)}\}_\infty}$$

is the defining equation of $J^{\infty,\infty}(E, n, n-1)$.

## DEFINITION

$J^{\infty,\infty}(E, n, n-1)$ is the diffiety of initial data.

# Structure of $J^{\infty,\infty}(E,n,n-1)$

From

$$\begin{tikzcd}
& J^\infty(J^\infty(E,n),n-1) & \\
& J^{\infty,\infty}(E,n,n-1) \arrow[u] \arrow[dl, "p"] \arrow[dr, "q"] & \\
J^\infty(E,n) & & J^\infty(E,n-1)
\end{tikzcd}$$

it looks evident that $J^{\infty,\infty}(E,n_2,n_1)$ possesses an inherited $(n-1)$–dimensional distribution $\mathcal{D}$, and two infinite–dimensional distributions. Denote by $\widetilde{\mathcal{C}}$ the one induced by $p^*$.

# Structure of $J^{\infty,\infty}(E,n,n-1)$

From

$$\begin{tikzcd} & J^\infty(J^\infty(E,n),n-1) & \\ & J^{\infty,\infty}(E,n,n-1) \arrow[u] \arrow[dl, "p"'] \arrow[dr, "q"] & \\ J^\infty(E,n) & & J^\infty(E,n-1) \end{tikzcd}$$

it looks evident that $J^{\infty,\infty}(E,n_2,n_1)$ possesses an inherited $(n-1)$–dimensional distribution $\mathcal{D}$, and two infinite–dimensional distributions. Denote by $\widetilde{\mathcal{C}}$ the one induced by $p^*$.

# Jet–prolongations

Any $n$–dimensional manifold $L$ produces the embedding $\widetilde{j}_\infty(L)$ of $J^\infty(L, n-1)$ into $J^{\infty,\infty}(E, n, n-1)$, which closes the diagram

$$\begin{array}{ccc} J^\infty(L, n-1) & \xrightarrow{\widetilde{j}_\infty(L)} & J^{\infty,\infty}(E, n, n-1) \\ \downarrow{\scriptstyle \pi_{\infty,0}} & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{j_\infty(L)} & J^\infty(E, n) \end{array}$$

# Lifted equation

Leaves of $\widetilde{\mathcal{C}}$ are precisely the embedded jet spaces $J^\infty(L, n_1)$, and as such are in one–to–one correspondence with the leaves of $\mathcal{C}$ in $J^\infty(E, n)$.
In other words, any equation $\mathcal{E}$ is equivalent to its own lifting $\widetilde{E} = p^*(\mathcal{E})$.

# Cohomology

Two non–trivial results can be proved:

- the $\mathcal{D}$–spectral sequence is one–line;
- the term $E_1$ of the $\widetilde{\mathcal{C}}$–spectral sequence is trivial above the line $q = n$

# Main conjecture

The following result, originally proposed by L. Vitagliano (unpublished note), has not yet been proved in general, but it holds true accordingly to C. Rovelli, in his paper *Covariant hamiltonian formalism for field theory: Hamilton-Jacobi equation on the space* $G$, `arXiv:gr-qc/0207043v2`, where $G$ is his own version of the space of initial data.

## CONJECTURE

Denote by $V\mathrm{sym}\,(\widetilde{\mathcal{E}_{\mathrm{EL}}})$ the Lie algebra of $p$–vertical $\infty$–esimal symmetries of $\widetilde{\mathcal{E}_{\mathrm{EL}}}$. Then

$$\mathcal{E}_{\mathrm{EL}} \cong \frac{\widetilde{\mathcal{E}_{\mathrm{EL}}}}{V\mathrm{sym}\,(\widetilde{\mathcal{E}_{\mathrm{EL}}})},$$

i.e., the space of trajectories of $V\mathrm{sym}\,(\widetilde{\mathcal{E}_{\mathrm{EL}}})$ is made of null directions of a suitable (secondary) symplectic 2–form.