1.On integrability of equations of mathematical physics

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Investigation of the equations of mathematical physics shown that equations describing any processes are not integrated without additional conditions. It follows from the functional relation that is derived from these equations. This relation proves to be nonidentical, and this fact points to the nonintegrability of the equations. However, under realization of the conditions of degenerate transformations, from the nonidentical relation it follows the identical one on some structure. This points out to the local integrability and realization of a generalized solution.

2. Such results have been obtained by using skew-symmetric forms, which are obtained from differential equations and, in distinction to exterior forms, are evolutionary ones and are defined on nonintegrable manifolds (such as tangent manifolds of differential equations, Lagrangian manifolds and so on), were used.

Three types of equations of mathematical physics, namely, the equations, which describe any physical processes, the equations of mechanics and physics of continuous media, and field-theory equations are analyzed.

3.The functional properties and integrability of equations or sets of equations depend on whether or not the derivatives of differential equations or the equation in the sets of differential equations are conjugated.

4. The equations, which describe any physical processes

Let's consider the elementary case: a first-order partial differential equation

(1) $$F(x^i, u, p_i) = 0, \quad p_i = \partial u / \partial x^i$$

- Let us consider the functional relation

(2) $$du = \theta$$

where $\theta = p_i d x^i$ is a skew-symmetric differential form of the first degree (the summation over repeated indices is implied).

The functional relation (2) have specific feature: in the general case, when differential equation (1) describes any physical processes, this relation is *nonidentical* one.

5. The left-hand side of this relation involves a differential, and the right-hand side isn't differential, as a differential form $\theta = p_i d x^i$ is an unclosed form . Without supplementary conditions, the differential this form is not equal to zero: $d\theta = d(p_i dx^i) = K_{ij} dx^i dx^j$, where a commutator

$$K_{ij} = \frac{\partial p^j}{\partial x^i} - \frac{\partial p^i}{\partial x^j} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \neq 0$$

(from equation (1) it does not follow (explicitly) that the mixed derivatives is permutation one).

The nonidentity of functional relation (2) points to a fact that without additional conditions the derivatives p_i of original equation do not make up a differential. This means that the corresponding solution u of the differential equation will not be a function of only variables x^i . The solution will depend on the commutator K_{ij} , that is, it will be a functional.

6. Thus one can see that the nonintegrability of differential equations is due to the nonconjugacy of the derivatives with respect to different variables : the commutator made up of relevant mixed derivatives is nonzero.
Without the realization of additional conditions, the solution will depend on this commutator, that is, it will be a functional.

7. **To obtain a solution that is a function (i.e., the derivatives of this solution make up a differential), it is necessary to add the closure condition for the form and for relevant dual form (the functional plays a role of a form dual) :**

$$\begin{cases} dF(x^i, u, p_i) = 0 \\ d(p_i dx^i) = 0 \end{cases}$$

8. **If we expand the differentials, we get a set of homogeneous equations with respect to dx^i and dp_i (in the $2n$-dimensional tangent space):**

$$\begin{cases} \left(\dfrac{\partial F}{\partial x^i} + \dfrac{\partial F}{\partial u} p_i \right) dx^i + \dfrac{\partial F}{\partial p_i} dp_i = 0 \\ dp_i dx^i - dx^i dp_i = 0 \end{cases} \qquad \textbf{(4)}$$

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- **It is well-known that** *vanishing the determinant* **(composed of coefficients at dx^i, dp_i) is a solvability condition of the system of homogeneous differential equations. This leads to relations:**

$$\frac{dx^i}{\partial F/\partial p_i} = \frac{-dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \qquad \textbf{(5)}$$

This relation specify the integrating direction - a pseudostructure, on which the form $\theta = p_i d x^i$ turns out to be closed one, i.e. is a differential. On the pseudostructure, which is defined by relation(5), the derivatives p_i of differential equation (1) constitute a differential $p_i d x^i = d_\pi u$ and this means that the solution of equation (1) becomes a function. *It is generalized solutions.*

9. This points out to the local (on the pseudostructures) integrability of the initial equation.

Similar functional properties have the solutions to all differential equations describing physical processes.

10. Thus one can see that the differential equations describing any physical fields can have solutions of two types, namely, generalized solutions which depend on variables only, and the solutions which are functionals since they depend on the commutator made up by mixed derivatives.

The dependence of the solution on the commutator may lead to instability. The instability develops when the integrability conditions are not realized and exact (generalized) solutions are not formatted. (Thus, the solutions to the equations of the elliptic type may be unstable.)

One can see that the qualitative theory of differential equations that solves the problem of unstable solutions and integrability bases on the properties nonidentical functional relation.

11. A specific feature of generalized solutions consists in the fact that they can be realized only under degenerate transformations. The relations (5) corresponding to generalized solutions had been obtained the condition of vanishing the determinant composed of coefficients at dx^i, dp_i in the set of equations (4). This is a condition of degenerate transformation. (They are connected with symmetries of commutators of skew-symmetric forms.)

12. Integrability of equations of mechanics and physics

Under description of mechanics and physics it is necessary to investigate the conjugacy of not only derivatives in different directions but also the conjugacy (consistency) of the equations. In this case from set of equations one also obtains nonidentical relation that allows to investigate integrability of equations and features of their solutions.

- Equations of mechanics and physics of continuous media are
- equations that describe the conservation laws for energy, linear
- momentum, angular momentum and mass.
- **Let us analyze the equations of energy and linear momentum.**

13. Evolutionary relation

In the accompanying reference system, which is tied to the manifold built by the trajectories of particles (elements of material system) the equations of energy and linear momentum are written in the form

- here ψ - the functional of the state (action functional, entropy, wave function are examples of the functional)

(1) $$\frac{\partial\psi}{\partial\xi^1} = \mathbf{A}_1$$

(2) $$\frac{\partial\psi}{\partial\xi^\nu} = \mathbf{A}_\nu \qquad \nu = 2,...$$

ξ^1 - coordinate along the trajectory

ξ^ν - coordinates normal to the trajectory

- A_μ - an expression that is depends on specific features of the system and on external energy actions onto the system.

Eqs. (1), (2) can be convoluted into the relation

- $$d\psi = A_\mu d\xi^\mu$$

14. This relation can be written as

$$d\psi = \omega$$

where $\omega = A_\mu d\xi^\mu$ - the differential form of the first degree.

In the general case **(for <u>energy,</u> linear momentum, angular momentum and mass)** this relation will be the form

$$d\psi = \omega^p, \quad p = 0,1,2,3$$

15. The relation obtained from the equation of the conservation laws possess the properties that enable one to investigate the integrability of the original set of equations.

16. Properties of evolutionary relation

1) Relation $d\psi = \omega$ is an *evolutionary relation*
since the equations are evolutionary ones.

2) Evolutionary relation is *a nonidentical relation*
as it involves an unclosed differential form ω **:** $d\omega \neq 0$
(this form is not differential as opposed to left-hand side of relation)

$$d\omega = K_{\alpha\beta} d\xi^{\alpha} d\xi^{\beta} \quad : \quad K_{\alpha\beta} = \left(\frac{\partial A_{\beta}}{\partial \xi^{\alpha}} - \frac{\partial A_{\alpha}}{\partial \xi^{\beta}}\right) \neq 0$$

commutator $K_{\alpha\beta}$ **of the form** ω **is nonzero as the coefficients** A_{μ} **of the form depend on energetic action and on force action which have different nature, and hence, the commutator constructed from derivatives of such coefficients cannot be equal to zero.**

3) Evolutionary nonidentical relation is *a selfvarying one* as it is an evolutionary relation and it contains two objects one of which appears to be unmeasurable and cannot be compared with another one, and therefore the process of mutual variation cannot terminate.

17. Nonidentity of the evolutionary relation, as well as nonidentity on functional relation (2), means that initial equations of balance conservation laws are not conjugated, and hence they are not integrable without additional conditions.

The solutions of these equations can be functional or generalized ones. In this case generalized solutions are obtained only under degenerated transformations.

18. The selfvarying evolutionary relation leads to realization of the conditions of **degenerate transformation.**

Under degenerate transformation from nonidentical relation is realized an closed form and the relation that is identical on pseudostructure is obtained. This points out to the local integrability of the initial equations.

19.The realization of exterior inexact form and dual form. Origination of differential-geometrical structures

Under degenerate transform it are realized conditions of closure of exterior and dual forms

$$d\omega \neq 0 \xrightarrow{\textit{degenerate transform}} \begin{cases} d_\pi \omega = 0 \\ d_\pi {}^*\omega = 0 \end{cases}$$

The relations obtained $d_\pi \omega = 0$, $d_\pi {}^*\omega = 0$
are closure conditions for exterior inexact form and for dual form.
This means that it is realized of pseudostructure (the closed dual form ${}^*\omega_\pi$) and formatting the closed inexact form ω_π .

It points to origination of differential-geometrical structures

$$\begin{cases} \omega_\pi \\ {}^*\omega_\pi \end{cases} \quad \textbf{a pseudostructure and a conservative quantity}$$

which has physical sense. These are structures from which physical fields are formatted. Such structures may be named as the physical structures.

20. Obtaining an identical relation from a nonidentical one

On the pseudostructure from evolutionary relation $d\psi = \omega$ follows identical relation $d_\pi \psi = \omega_\pi$ (closed exterior form is differential: $\omega_\pi = d_\pi \vartheta$, therefore on the right and at the left there are differentials) .

The identity of the relation means that on pseudostructures the equations of conservation laws become consistent and integrable. The realization of identical relation means that the state differential (the left-hand part of identical relation) is realized. This point to availability of the state function that specifies in realization generalized (only on pseudostructure) solution of the initial equations. This points out to the local (on the pseudostructures) integrability of the initial equations.

21. **Pseudostructures constitute the integral surfaces (such as characteristics, singular points, potentials of simple and double layers and others) on which the quantities of material system desired (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). This are generalized solutions.**

One can see that the integral surfaces are obtained from the condition of degenerate transformation of the evolutionary relation. The conditions of degenerate transformation are a vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues and others. They are connected with the symmetries, which can be due to the degrees of freedom (for example, the translational, rotational and oscillatory freedom of the of material system).

22. The degenerate transformation is realized as the transition from nonintegrable manifold (for example, tangent one) to the integrable structures and surfaces (such as the characteristics, potential surfaces, eikonal surfaces, singular points).

Mathematically to it there corresponds transition one frame of reference to another nonequivalent frame of reference.

Thus, one can see that the problem of integrability is based on the properties of nonidentical evolutionary relation.

23. Nonidentical evolutionary relation, which is obtained from the equations for material media, allows to understand the specific features of the field-theory equations as well.

One can see that there exists the correspondence between the field-theory equations and the nonidentical evolutionary relation.

24. Functional properties of the field-theory equations

- The field-theory equations differ in their functional properties
- from the equations for material systems. The equations for
- material systems are ordinary differential equations. The
- solutions to these equations are **functions** (which describe
- physical quantities such as a velocity, pressure and density). And
- the solutions to the field-theory equations are **differentials**
- because these equations must describe physical structures,
- which the closed inexact (on pseudostructure) exterior forms are assigned. Only the equations that have the form of relations (nonidentical) may have the solutions which are differentials rather then functions.

25. One can verify that all equations of existing field theories have the form of nonidentical relations in differential forms or in the forms of their tensor or differential (i.e. expressed in terms of derivatives) analogs.

--The Einstein equation is a relation in differential forms.

--The Dirac equation relates Dirac's { bra-} and { cket}- vectors, which made up the differential forms of zero degree.

--The Maxwell equations have the form of tensor relations.

--The field equation and Schridinger's one have the form of relations expressed in terms of derivatives and their analogs.

The identical relations, which include closed exterior forms or their tensor or differential analogs, are the solutions to such field-theory equations.

26. Another specific feature of the field-theory equations consists in the fact that all field-theory equations are nonidentical relations for functionals such as a wave function, action functional, Einstein's tensor and so on. The Pointing vector is such a functional for the equations of electromagnetic field. (Entropy is such a functional for the fields generated by thermodynamical and gas-dynamical systems.)

The evolutionary relation obtained from the equations for material systems is a nonidentical relation for all these functionals. That is, all field-theory equations are an analog to the evolutionary relation.

27. The correspondence between the field-theory equations and the evolutionary relation points to a connection between field theories and the equations for material systems. This means that the equations of conservation laws for material systems are the basis of field theories.

It can help in solving the problems of existing field theories and building the general field theory.